Dynamic Covariate Balancing: Estimating Treatment Effects over Time with Potential Local Projections

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Two Examples

Ex1 "Effect of negative advertisement on election outcome?" (Blackwell, 2013)

- Dynamic choice of which advertisement to send;
- Politicians may make strategic choices based on past polls.

Ex2 "What is the effect of democracy on economic growth?" (Acemoglu et al., 2019)

- Hard question.
- Long and short term impacts may be different;
- Past growth may determine the selection into treatment;
- It is necessary to control for many potential confounders.

Inference on Treatment Histories

Suppose we collect data from an observational study with T periods and n individuals.

Goal: Inference on the effect of a treatment history.

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Goal: Inference on the effect of a treatment history.

Challenges:

- Individuals select dynamically on arbitrary past information;
- Selection mechanism (propensity score) hard to estimate/unknown;
- Intermediate covariates and outcomes depend on past treatment assignments;
- Many potential confounders (high-dimensional covariates).

Content

Data and Notation

Ex-post evaluation:

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Data:

$$\Big(X_{i,1}, X_{i,2}, \cdots, Y_{i,1}, \cdots, Y_{i,T}, D_{i,1}, \cdots, D_{i,T}\Big)_{i=1}^{n} \sim_{i.i.d.} \mathcal{P}$$

- $X_{i,t} \in \mathcal{X}_t$ - time-varying covariates;
- $Y_{i,t} \in \mathcal{Y}$ - intermediate outcomes;
- $D_{i,t} \in \{0,1\}$ - dynamic treatments;

$\Rightarrow$ Potential outcomes: $Y_{i,t}(d_1, \cdots, d_t)$.

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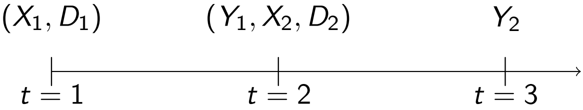
Goal

- Inference on treatment effects at time T.

Illustration: Two Periods



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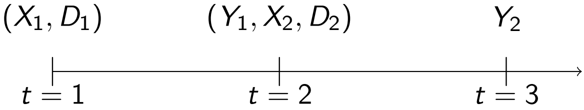

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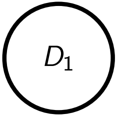

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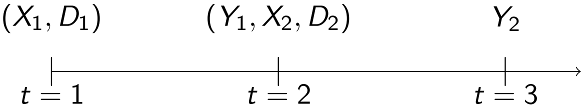

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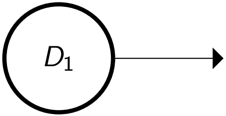

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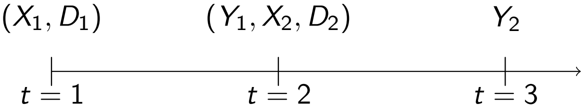

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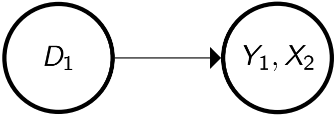

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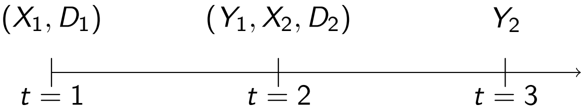

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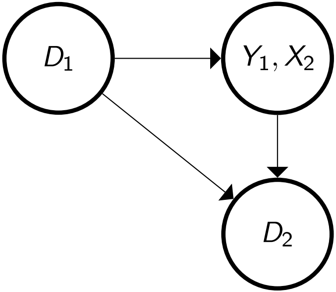

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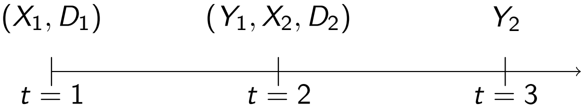

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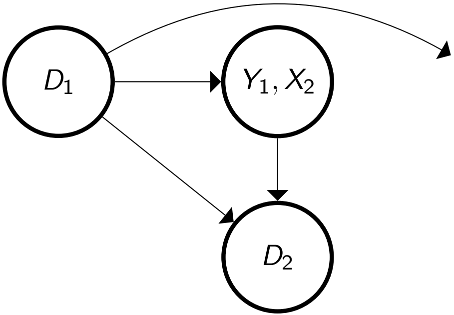

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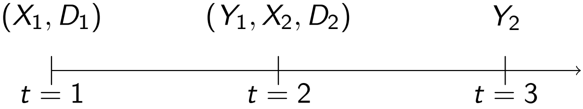

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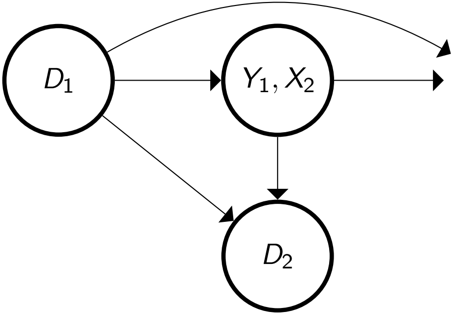

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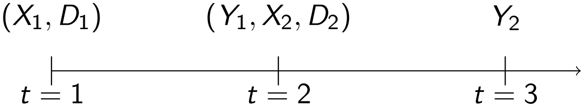

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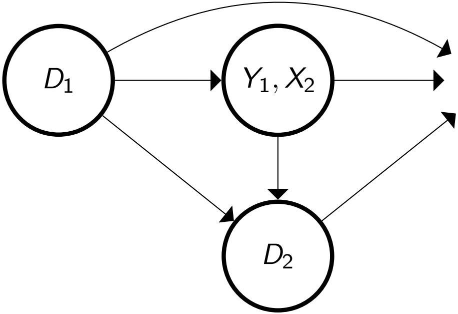

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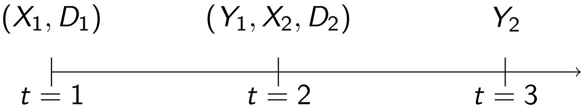

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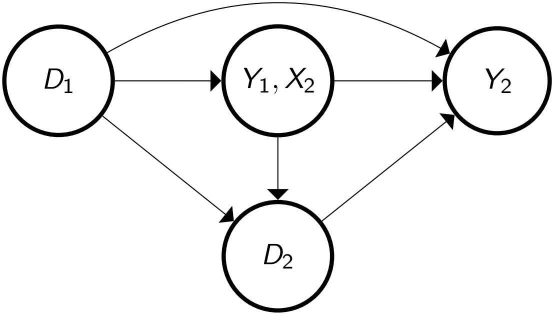

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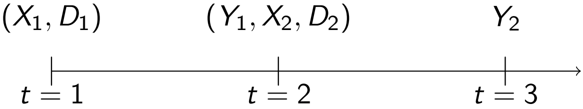

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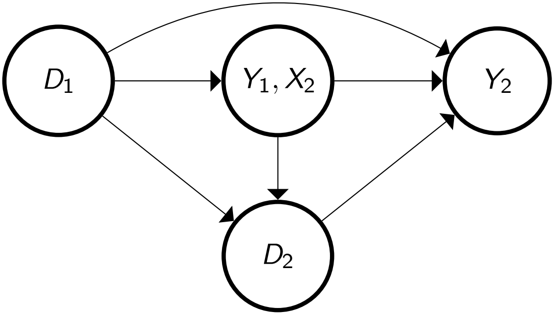

Goal Inference on $\mathbb{E}\Big[Y_2(1, 0) - Y_2(0, 0)\Big]$ (or conditional on baselines).

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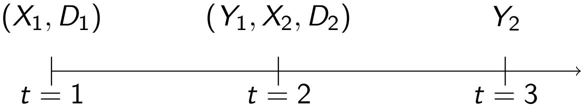

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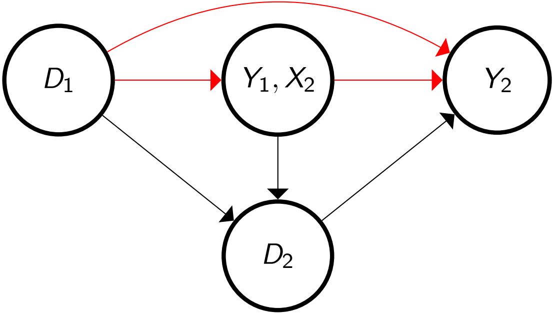

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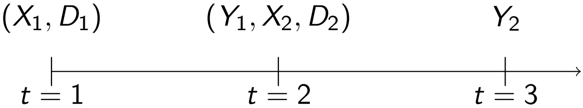

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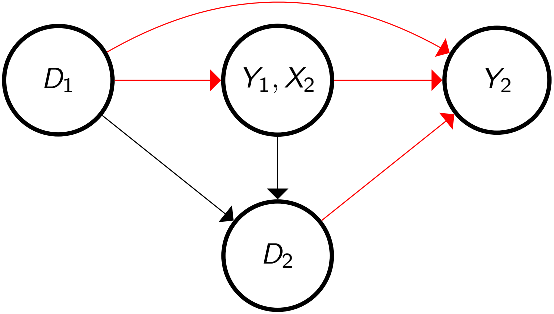

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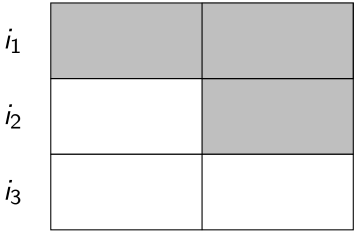

i_1 $\Big(Y_1(0), X_2(0), Y_2(0,0)\Big)$?;

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i_3 ...

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Review: some intuitive estimators

1. Regress end-line outcomes on covariates:

$$Y_{i,T} = \sum_{t=0}^{T-1} \alpha_t D_{i,t} + X_{i,t}\beta_t + \sum_{t=1}^{T-1} Y_{i,t}\phi_t + \varepsilon_{i,T}.$$

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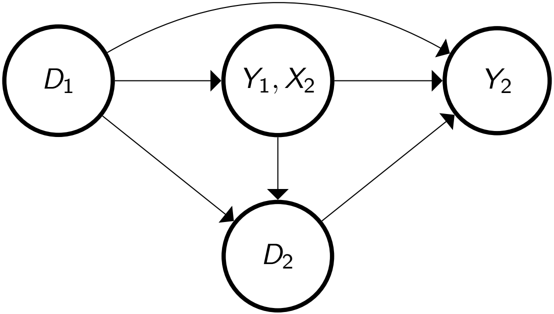

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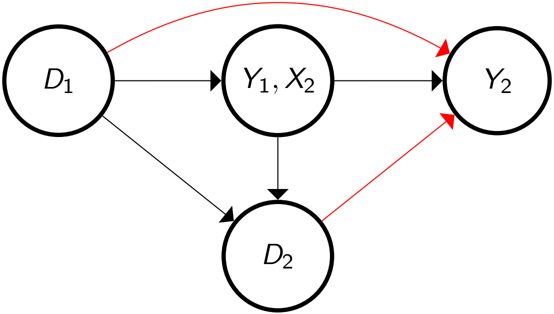

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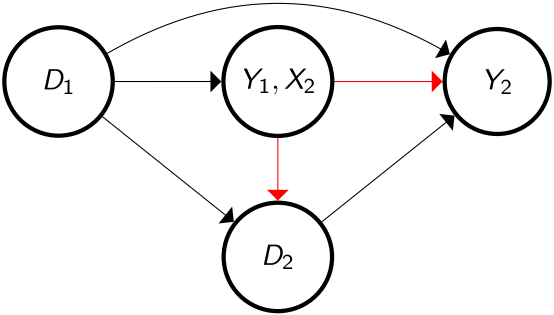
$\Rightarrow$ omitted variable bias.



Review: some intuitive estimators

3. Run separate regressions for

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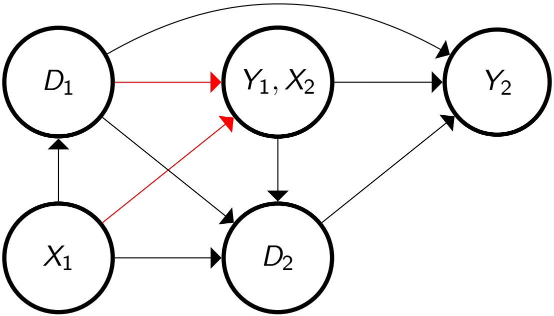
Prone to large estimation error in high dimensions.



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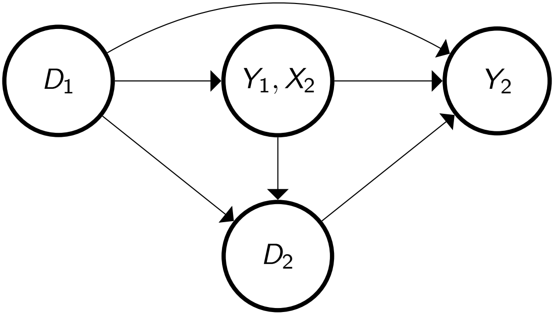

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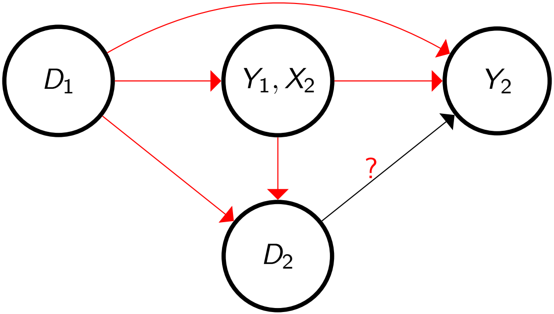

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   - ⇒ It does not require to specify and estimate the propensity score;
   - ⇒ It guarantees an asymptotic vanishing bias in high dimensions;
   - ⇒ More stable than (A)IPW estimators in the presence of poor overlap;
   - ⇒ Generalizes Zubizarreta (2015) and Athey et al. (2018) to dynamics.

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Conditions:

1. Sequential ignorability:
   - ⇒ Treatments are assigned sequentially based on past observations and exogenous unobservables.
2. Approximate linearity:
   - ⇒ Linearity of outcomes on past outcomes and high-dimensional covariates.

Related Literature

- Dynamic Treatments' literature and marginal structural models [E.g., Robins, 1986, Robins et al., 2000, Bang and Robins 2005, Boruvka et al., 2018; Blackwell, 2013; Bojinov and Shephard 2019; Lewis and Syrgkanis, 2020; Bodory et al. 2020]
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- Estimation and inference in time-series/(macro)econometrics [Jorda, 2005; Stock and Watson, 2018, Angrist et al. (2018); Rambachan and Shephard (2019)]
  - $\Rightarrow$ Exogeneity and independence of shocks (treatments).
- Balancing in *i.i.d.* settings [E.g., Zubizarreta (2015); Athey et al. (2018)]; difference-in-differences, synthetic controls, panel data [E.g., Ben-Michael et al.; Athey and Imbens (2022)); Arkhangelsky and Imbens (2019); ...]

$\Rightarrow$ Here estimation and inference under sequential exogeneity.

Content

Basic Model: Example

Basic model that we generalize in the following slides:

$$Y_{i,2} = \Big[Y_{i,1}, X_{i,1}, D_{i,1}, D_{i,2}\Big]\beta_2 + \varepsilon_{i,2}, \quad \varepsilon_{i,2} \perp D_{i,2} \Big| D_{i,1}, X_{i,1}, Y_{i,1}$$
$$Y_{i,1} = \Big[X_{i,1}, D_{i,1}\Big]\beta_1 + \varepsilon_{i,1}, \qquad (\varepsilon_{i,1}, \varepsilon_{i,2}) \perp D_{i,1} \Big| X_{i,1}.$$

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\end{aligned}
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We can write

$$
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Y_{i,2}(d_1, d_2) &= \Big[Y_{i,1}(d_1), X_{i,1}, d_1, d_2\Big]\beta_2 + \varepsilon_{i,2} \\
Y_{i,1}(d_1) &= \Big[X_{i,1}, d_1\Big]\beta_1 + \varepsilon_{i,1}
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$$Y_{i,1} = \Big[X_{i,1}, D_{i,1}\Big]\beta_1 + \varepsilon_{i,1}, \quad (\varepsilon_{i,1}, \varepsilon_{i,2}) \perp D_{i,1}\Big| X_{i,1}.$$

We can write

$$Y_{i,2}(d_1, d_2) = \Big[Y_{i,1}(d_1), X_{i,1}, d_1, d_2\Big]\beta_2 + \varepsilon_{i,2}$$

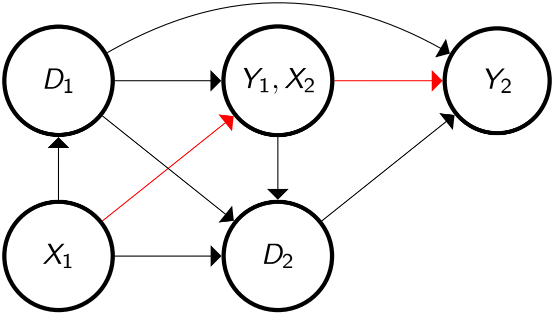
$$Y_{i,1}(d_1) = \Big[X_{i,1}, d_1\Big]\beta_1 + \varepsilon_{i,1}$$

$\Rightarrow$ $Y_{i,2}(d_1, d_2)$ is linear in $X_{i,1}$ *unconditionally* on $Y_{i,1}(d_1)$:

$$Y_{i,2}(d_1, d_2) = \Big[1, 1, X_{i,1}\Big]\beta_{d_1,d_2} + \nu_{i,1}^{d_1,d_2}, \quad \mathbb{E}\Big[\nu_{i,1}^{d_1,d_2}\Big| X_{i,1}, D_{i,1}\Big] = 0.$$

Example: Illustration

- Model on potential outcomes: linear dependencies?



- Remaining components are left unspecified.

Basic Model: Example

Recall the starting model:

$$Y_{i,2} = \Big[Y_{i,1}, X_{i,1}, D_{i,1}, D_{i,2}\Big]\beta_2 + \varepsilon_{i,2}, \quad \varepsilon_{i,2} \perp D_{i,2}\Big|D_{i,1}, X_{i,1}, Y_{i,1}$$

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$\Rightarrow$ Model on the observed outcomes also depend on $(X_1, D_1) \to D_2$;

$\Rightarrow$ Model on potential outcomes is more flexible.

Model Specification: General Case

Define $H_{i,2}(d_1) = \Big[X_{i,1}, X_{i,2}(d_1), Y_{i,1}(d_1)\Big]$ (also with intercepts).

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Sequential ignorability:

$$Y_{i,2}(d_1, d_2) \perp D_{i,2}\Big|H_{i,2}, D_{i,1}, \quad \Big(Y_{i,2}(d_1, d_2), H_{i,1}(d_1)\Big) \perp D_{i,1}\Big|X_{i,1}.$$

Identification and Estimation of the Linear model

! $\mathbb{E}\Big[Y_{i,2}\Big|X_{i,1}, D_{i,1} = 1, D_{i,2} = 1\Big] \neq \mathbb{E}\Big[Y_{i,2}(1,1)\Big|X_{i,1}\Big]$

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Coefficients' Estimation (with Lasso):

$$Y_2 \rightarrow \Big[X_2, X_1, D_1=d_1, D_2=d_2\Big] \rightarrow H_2\hat{\beta}^2_{d_1,d_2}$$
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- Problem: bias due to high-dimensionality;
- Unknown propensity score: (a)IPW can be prone to misspecification and poor overlap

Estimating $\mathbb{E}[Y_2(1,1)]$:

	$D_2 = 1$	$D_2 = 0$
$D_1 = 1$		
$D_1 = 0$		

Balancing conditions:

$$\hat{\gamma}_1 : \left|\left|\bar{X}_1 - \hat{\gamma}_1^\top X_1\right|\right|_\infty \leq \delta_n, \quad \hat{\gamma}_2 : \left|\left|\hat{\gamma}_2^\top H_2 - \hat{\gamma}_1^\top H_2\right|\right|_\infty \leq \delta_n$$

Balancing: Illustration

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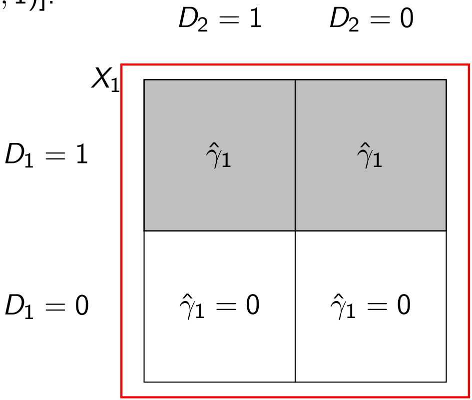

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H_2

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Estimators

- ⇒ Balancing to guarantee neglegible ($o_p(1/\sqrt{n})$) estimation error;
- ⇒ It does not require the specification (and estimation) of the propensity score.

Two periods: $\hat{\mu}_{1,1} = \hat{\gamma}_2^\top \underbrace{(Y_2 - H_2\hat{\beta}^2_{1,1})}_{Y_2 - Pred2} + \hat{\gamma}_1^\top \underbrace{(H_2\hat{\beta}^2_{1,1} - X_1\hat{\beta}^1_{1,1})}_{Pred2 - Pred1} + \underbrace{\bar{X}_1\hat{\beta}^1_{1,1}}_{Est1}$

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Step t (Sequential)

$$\hat{\gamma}_t = \text{argmin}_{\gamma_t} ||\gamma_t||^2$$

$$\Big|\Big| \sum_{i=1}^n \hat{\gamma}_{i,t-1} H_{i,t} - \sum_{i=1}^n \gamma_{i,t} H_{i,t} \Big|\Big|_\infty \le \delta(n,p), \quad \sum_{i=1}^n \gamma_{i,t} = 1 .$$

$$||\gamma_t||_\infty \le \log(n) n^{-2/3}, \quad 0 \le \gamma_{i,t} \le \prod_{s=1}^t 1\{D_{i,s} = d_s\}$$

Rationale

Lemma 1

For any $\hat{\gamma}_1$, and $\hat{\gamma}_2$, where $\hat{\gamma}_{i,2} = 0$ if $(D_{i,1}, D_{i,2}) \neq (1,1)$, we have

$$\hat{\mu}(1,1) - \mu(1,1) = \underbrace{(\beta^1_{1,1} - \hat{\beta}^1_{1,1})^\top \left(\bar{X}_1 - \hat{\gamma}_1^\top X_1\right)}_{BIAS1} + \underbrace{(\beta^2_{1,1} - \hat{\beta}^2_{1,1})^\top \left(\hat{\gamma}_2^\top H_2 - \hat{\gamma}_1^\top H_2\right)}_{BIAS2}$$
$$+ \underbrace{\hat{\gamma}_2^\top \left(Y_2 - H_2\beta^2_{1,1}\right)}_{\text{Error 1}} + \underbrace{\hat{\gamma}_1^\top H_2\beta^2_{1,1} - \hat{\gamma}_1^\top X_1\beta^1_{1,1}}_{\text{Error 2}}. \quad (1)$$

⇒ Balancing control Bias1 and Bias2. Bias2 arises due to dynamics

? Why should we estimate γ_1, γ_2 sequentially?

Residuals from balancing

Lemma 2

Let the sigma algebra $\sigma(\hat{\gamma}_1) \subseteq \sigma(X_1, D_1)$ and $\hat{\gamma}_{i,1} = 0$ if $D_{i,1} \neq 1$. Then

$$\mathbb{E}\Big[\hat{\gamma}_{i,1} H_{i,2} \beta^2_{1,1} - \hat{\gamma}_{i,1} X_{i,1} \beta^1_{1,1} \Big| X_{i,1}, D_{i,1}\Big] = 0.$$

Intuition:

- If $D_{i,1} \neq 1$ then the expression is zero by construction;
- If $D_{i,1} = 1$, we can use the law of iterated expectations;
- If $\hat{\gamma}_1$ depends on future observations, or $\hat{\gamma}_{i,1} = 1$ for untreated units, the expression is not necessarily zero.

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Under regularity assumptions:

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$\Rightarrow$ Confidence with Gaussian critical values if $\sqrt{n}||\hat{\gamma}_t||_2$ converges almost surely (e.g., for Bernoulli design).

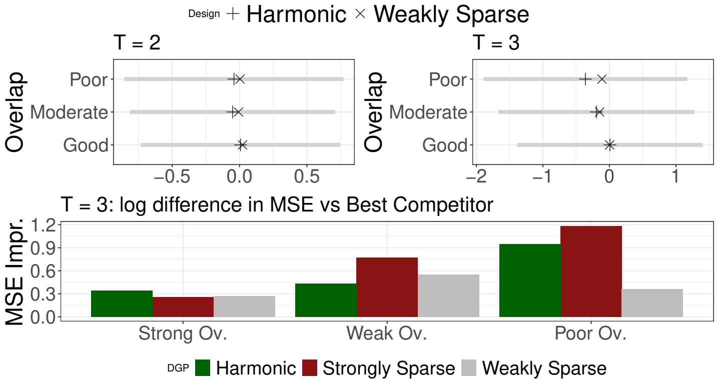
Remarks

- Trade-offs for δ_n: larger δ_n guarantees feasibility, but increases bias;
- Bias is of order $||\hat{\beta} - \beta||_1 \delta_n$. Longer T implies larger estimation error unless we assume limited carry-over effects;
- In the paper: algorithmic procedure to find the smallest feasible δ_n.

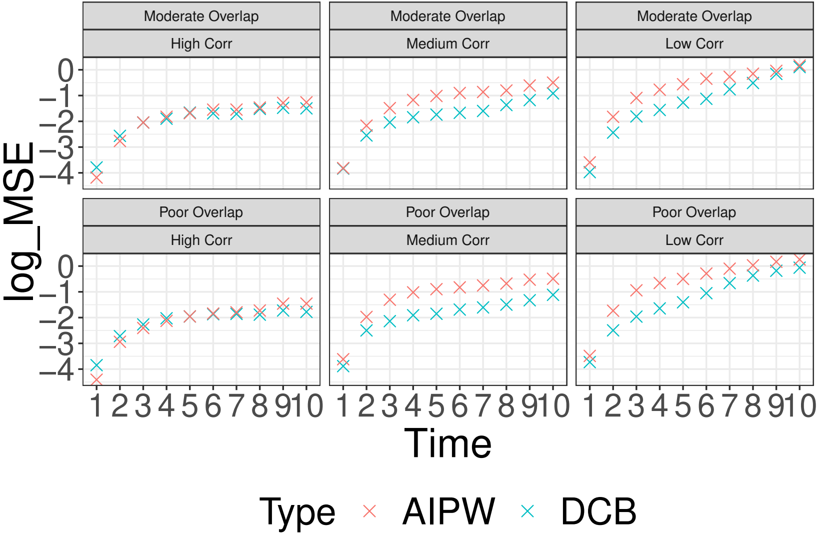
Content

Numerical Study

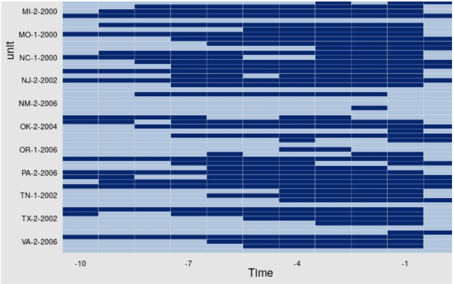
- Designs: $\beta^{(j)} = 1/j, \beta^{(j)} = 1/j^2$
- Linear model, different level of overlap.



Comparison with AIPW

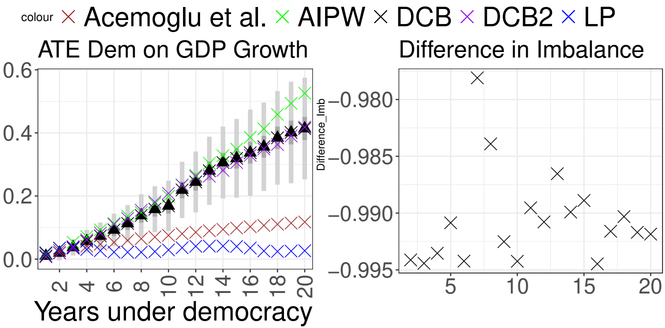


Advertisment and Elections



	ATE DCB	ATE AIPW
S1	-1.767	-0.57
	(0.70)	(1.45)
S2	-0.493	-0.22
	(0.764)	(1.33)

Democracy and Growth



Content

Conclusion

- We propose a method for estimating dynamic causal effects;
- We provide an estimation and inferential procedure using novel covariate balancing conditions;
- We characterize asymptotic properties of the estimator, and study its finite sample properties in numerical studies and empirical applications.

Questions?

Link `dviviano.github.io/projects`