Causal clustering: design of cluster experiments under network interference

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Example

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$\Rightarrow$ Challenge: information spillovers betw/ friends

$\Rightarrow$ Cluster designs (Karrer et al., 2022; Duflo and Banerjee, 2018)

(i) Researchers partition the population into subgroups (e.g., villages)

(ii) They randomize treatments at the cluster level

(iii) Estimator as difference in means betw/ treated and controls

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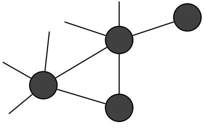

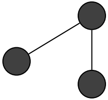

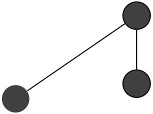

Cluster 1 Cluster 2 Cluster 3

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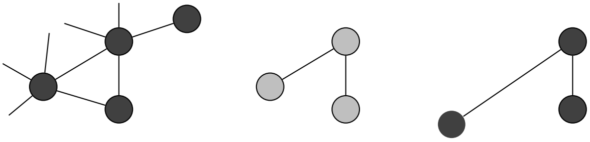

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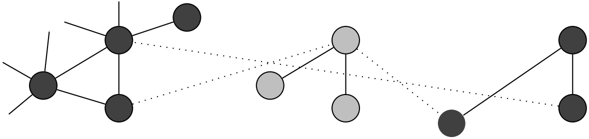

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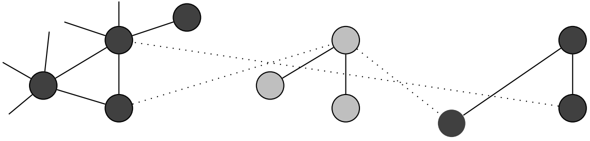

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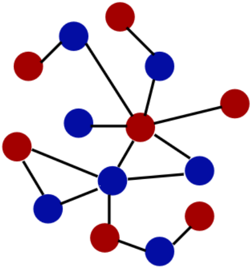
In practice...

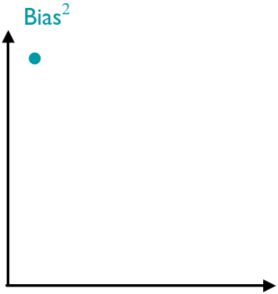
⇒ Not many clusters available (e.g., only one network)

⇒ Uncertainty over spillover (and direct) effects

⇒ Trade-offs in the choice and number of the clusters

Bias-variance tradeoff





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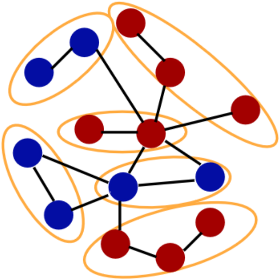

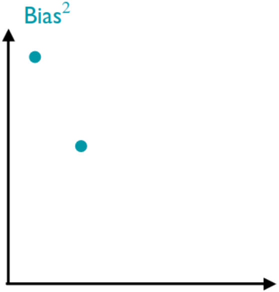

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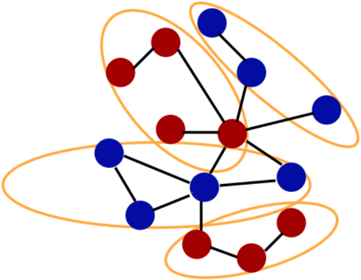

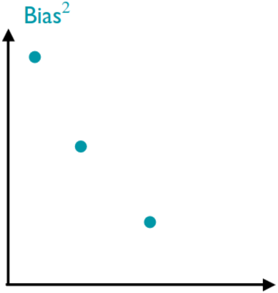

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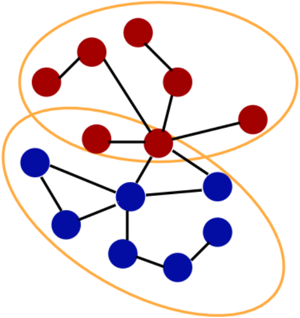

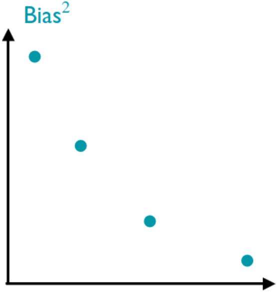

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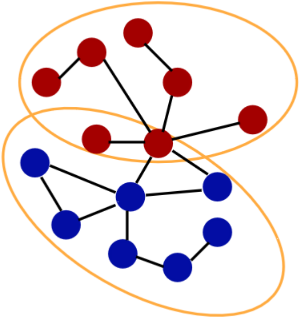

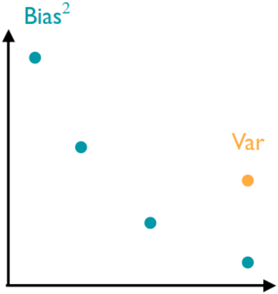

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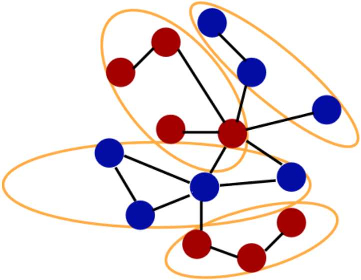

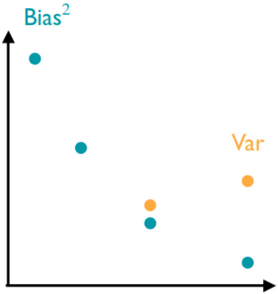

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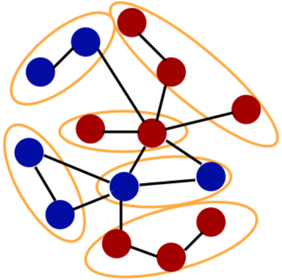

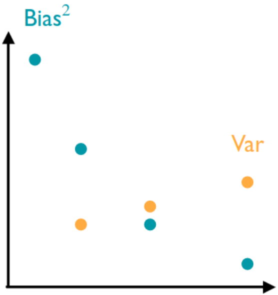

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  - Comparison with Bernoulli (*i.i.d.*) design with worst-case MSE
- How should you design the clusters?
  - MSE-Optimal clustering as simple optimization (sd program)
  - Application to unique data from Facebook graph
- Conceptual contribution: clustering for treatment effect estimation
  - Provide intuitive “cluster purity” measure as bias justification
- Theoretical contribution: model spillovers through local asymptotics
  - Variance only depends on average cluster size variation

Literature

- Experimental design with spillovers [Baird et al., 2018; Basse and Airoldi, 2018; Johari et al. (2020); Viviano (2020); Pouget-Abadie et al. (2019, 2023) ...]
- Inference under interference [Hudgens and Halloran (2008); Aronow and Samii (2017); Leung (2020); Athey et al. (2018); Savje et al. (2021); Liu and Wager, 2022]
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- Clustering in stat [Von Luxburg, 2007; Newman 2013; Lei et al. (2020), et al.]

  $\Rightarrow$ Here different focus on clustering for causal inference

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- Finite pop: n units connected under adjacency matrix A (observed)
  - Potential outcomes: $Y_i(\mathbf{d}), \mathbf{d} \in \{0,1\}^n$

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Model on spillover effects

- Class of models: for $\mathbf{d}' \in \{\mathbf{1}, \mathbf{0}\}$

$$\sup_{\mu_i(1,\cdot)\in\mathcal{M}_1} \Big|\mu_i(1,\mathbf{d}) - \mu_i(1,\mathbf{d}')\Big| = \frac{\bar{\phi}_n}{|\mathcal{N}_i|} \sum_{i\in\mathcal{N}_i} \Big|\mathbf{d}'_k - \mathbf{d}_k\Big|$$

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$\Rightarrow$ signal to noise is of order smaller than $n^{1/2}$
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E.g. Write the overall effect as $\tau_n = \Delta_n + \bar{\phi}_n$, for $\Delta_n, \bar{\phi}_n = o(1)$.

E.g. Information experiments [Karrer et al. (2021)]

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$\Rightarrow$ More clusters' size variation imply smaller variance

Worst-case variance/bias?

Lem Worst-case outcome model justifies cluster purity:

$$\sup_{\mu \in \mathcal{M}} \Big(\mathbb{E}\Big[\hat{\tau}_n | \mathbf{A}, \mathcal{C}_n\Big] - \tau_n\Big)^2 = \frac{\bar{\phi}_n}{n} \sum_{i=1}^{n} \frac{1}{|\mathcal{N}_i|} \underbrace{\Big|\mathcal{N}_i \bigcap \Big\{j : c(i) \neq c(j)\Big\}\Big|}_{\#i\text{'s friends not in } i\text{'s cluster}}$$

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Thm For $\mathcal{C}_{0,n}$ denoting a Bernoulli design, under regularities

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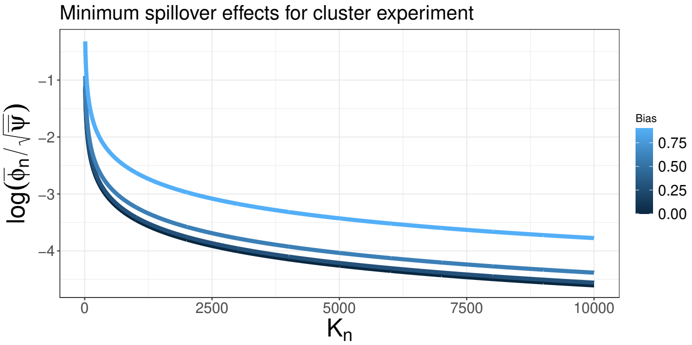

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Description	Mathematical Formulation	Implication
Spillovers are small and # of clusters is small	$\sqrt{K_n}\bar{\phi}_n = o(1)$	Run Bernoulli design

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Description	Mathematical Formulation	Implication
Spillovers are non-negligible and # of clusters is large	$\sqrt{K_n}\bar{\phi}_n \to \infty$	Run Cluster design

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Description	Mathematical Formulation	Implication
Spillovers are small and # of clusters is very large	$K_n \propto n, \bar{\phi}_n \propto n^{-1/3}$	Run Cluster design

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Description	Mathematical Formulation	Implication
Spillovers are not that small and # of clusters is very small	$K_n = \mathcal{O}(1), \bar{\phi}_n = o(1)$	Run Bernoulli design

Content

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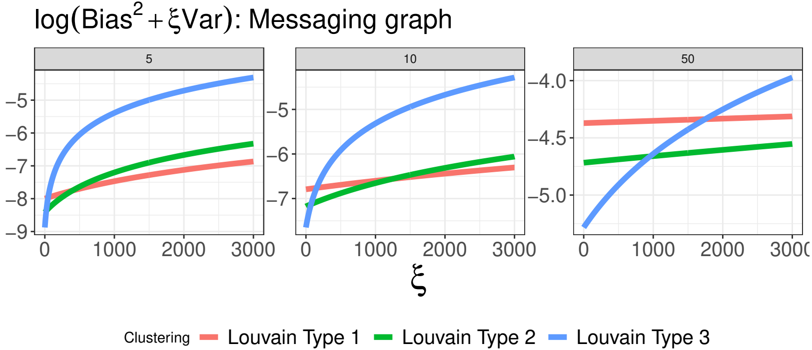

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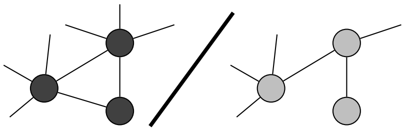

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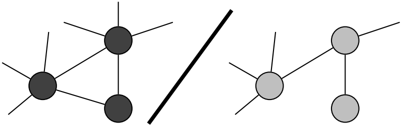

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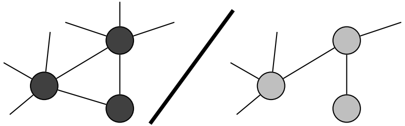

Thm Let $\mathbf{M}_{c,K}^{(i,k)} = 1\{c(i) = k\}$. For $K_n = K$, optimization is equivalent to

$$\max_c \mathrm{tr}\Big((\mathbf{L} - \frac{\xi_n'}{n}\mathbf{1}_n\mathbf{1}_n)\mathbf{M}_{c,K}\mathbf{M}_{c,K}^\top\Big), \quad \mathbf{L} = \mathrm{diag}(\mathbf{A}\mathbf{1}_n)^{-1}\mathbf{A}$$

Implementation

- Choose $n\xi_n'$ (e.g., 3.29)
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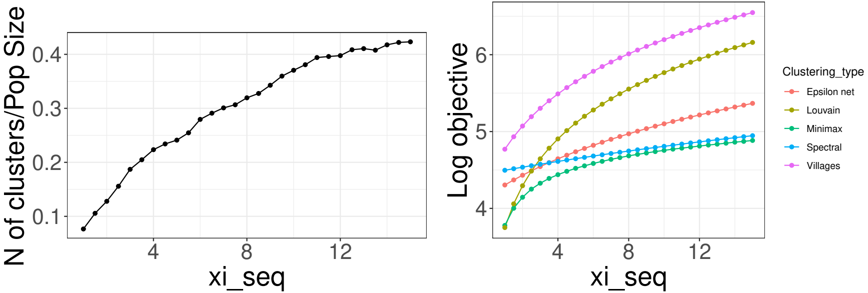

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Conclusions

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Thanks very much, questions? :)

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