Appendix: Policy Targeting under Network Interference

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The online Appendix contains additional extensions (Appendix B), a numerical study (Appendix C), and derivations (Appendices D). Appendix A at the end of the main text contains a complete description of the algorithms.

Appendix B Additional extensions

B.1 Constraints on Π_n that depend on D

Following Remark 3, in this subsection, I discuss a policy-function class

$$\tilde{\Pi}_n = \Big\{\tilde{\pi} : \mathcal{X} \times \{0,1\} \mapsto \{0,1\}, \tilde{\pi}(x,d) = \pi(x)(1-d) + d, \pi \in \Pi\Big\}, \tag{B.1}$$

for Π with finite VC dimension. Here $\tilde{\pi}(D_i, X_i)$ is one almost surely if the treatment in the experiment is one ($D_i = 1$). I define e, m^c as in Equation (9), here functions of $\tilde{\pi}$.[1]

Proposition B.1. *Let Assumptions 2.1, 2.2, 2.3, 2.5, and 3.1 hold. Consider a policy class* $\tilde{\pi}(X_i, D_i), \tilde{\pi} \in \tilde{\Pi}_n$, *with* $\tilde{\pi}^*_{m^c,e} \in \arg\max_{\tilde{\pi} \in \tilde{\Pi}_n} W_n(\tilde{\pi}, m^c, e)$. *For a universal constant* $\bar{C} < \infty$,

$$\mathbb{E}\Big[\sup_{\pi \in \tilde{\Pi}_n} W_{A,Z}(\pi) - W_{A,Z}(\tilde{\pi}^*_{m^c,e})\Big| A, Z\Big] \le \bar{C}\frac{\Gamma \mathcal{N}_n^{3/2}}{\gamma \delta_n}\sqrt{\frac{\log(\mathcal{N}_n)\mathrm{VC}(\Pi)}{n_e}}.$$

See Appendix D.4.2 for the proof. Proposition B.1 extends our results for policies constrained to always assign treatments to the treated individuals in the experiment.

B.2 Estimation error of nuisance functions with Algorithm 2

This section examines the estimation error $\sqrt{\mathcal{R}_n(A,Z) \times \mathcal{B}_n(A,Z)}$ in Theorem 3.3. Consider estimating $m(\cdot)$ with Algorithm 2. Algorithm 2 first partitions the units into K^* groups. Within each group, it constructs J equally sized folds. For two units (i, v), define $\phi_v^m(i) \in \{0,1\}$ with $\phi_v^m(i) = 1$ if *all* of the following conditions hold unit v is sampled ($R_v = 1$); v is in the *same* partition $k \in \{1, \cdots, K^*\}$ of i; and v is in any fold except the one containing unit i.[2] The effective sample size for estimation of $\hat{m}^{(i)}$ is $\sum_{v=1}^n R_v \phi_v^m(i)$ because, Algorithm 2

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[1]For example, for i, the propensity score is $e(\tilde{\pi}(X_i, D_i), T_i(\tilde{\pi}), Z_i, Z_{k\in N_i}, R_{k \in N_i}, |N_i|), \tilde{\pi} \in \tilde{\Pi}_n$.

[2]Following Algorithm 2's definitions, $\phi_v^m(i) = 1\{v \in (F_k^j)_{j=1}^J \setminus F_k^{j(i)}, k \text{ such that } i \in \cup_j F_k^j\}$.

uses sampled units not in the same fold of i, but in its same partition k. Define $\phi_v^e(i) \in \{0,1\}$, with $\phi_v^e(i) = 1$ if *all* of the following conditions hold: (a) unit v is sampled or, if not sampled, one of its friends is sampled ($R_v = 1$ or $(1-R_v)R_v^f = 1$); (b) v is in the *same* partition $k \in \{1, \cdots, K^*\}$ of i; and (c) v is in any fold except the one containing unit i, once we run Algorithm 2 to estimate $e(\cdot)$. Let $m \in \mathcal{M}, e \in \mathcal{E}$, for function classes $\mathcal{M}, \mathcal{E}$, and assume

$$\begin{aligned} \mathcal{R}_n(A,Z) &= \mathcal{O}\Big(\frac{1}{n}\sum_{i=1}^n C_{\mathcal{M}}\mathbb{E}\Big[\Big(1+\sum_{v=1}^n R_v\phi_v^m(i)\Big)^{-2\zeta_m}\Big|R_i=1,A,Z\Big]\Big) \\ \mathcal{B}_n(A,Z) &= \mathcal{O}\Big(\frac{1}{n}\sum_{i=1}^n \frac{1}{\delta_n^2}C_{\mathcal{E}}\mathbb{E}\Big[\Big(1+\sum_{v=1}^n R_v\phi_v^e(i)\Big)^{-2\zeta_e}\Big|R_i=1,A,Z\Big]\Big) \end{aligned} \tag{B.2}$$

for some $1/2 \geq \zeta_m, \zeta_e > 0$, and $C_{\mathcal{M}}, C_{\mathcal{E}}$ capturing the complexity of the function class (e.g., the sparsity parameter and number of coefficients in a Lasso regression, see Negahban et al. 2012). Here, ζ_m characterizes the convergence rate of the conditional mean function on a sample of *independent* units (by Algorithm 2), with $\Big(1+\sum_{v=1}^n R_v\phi_v^m(i)\Big)$ denoting the effective sample size to estimate $\hat{m}_i$. Similarly, ζ_e for the propensity score. I rescale the rates for the propensity score by $1/\delta_n^2$ because the propensity score is bounded from zero by δ_n.[3]

Equation (B.2) also captures the contribution to the estimation error of those units i belonging to groups with a few (finite number of) observations (see Algorithm 2).[4]

Proposition B.2. *Suppose the conditions in Theorem 3.3 and Equation* (B.2) *hold, and* $n_e = \alpha n, \alpha \in (0,1)$. *Then* $\sqrt{\mathcal{R}_n(A,Z)\times\mathcal{B}_n(A,Z)} = \mathcal{O}\left(\frac{\mathcal{N}_n^2 C_{\mathcal{M}}^{1/2}C_{\mathcal{E}}^{1/2}}{\delta_n n_e^{\zeta_m+\zeta_e}}\right)$. *In addition, if* $\mathcal{N}_n^{1/2}C_{\mathcal{M}}^{1/2}C_{\mathcal{E}}^{1/2}/n_e^{\zeta_m+\zeta_e} = \mathcal{O}\left(n_e^{-1/2}\right)$, *then* $\mathbb{E}\Big[\sup_{\pi\in\Pi_n} W_{A,Z}(\pi) - W_{A,Z}(\hat{\pi}_{\hat{m},\hat{e}})\Big|A,Z\Big] = \mathcal{O}\left(n_e^{-\xi}\right)$.

See Appendix D.4.3 for the proof. Proposition B.2 characterizes the rate of the estimation error. Here, $\mathcal{N}_n^{1/2}C_{\mathcal{M}}^{1/2}C_{\mathcal{E}}^{1/2}/n_e^{\zeta_m+\zeta_e} = \mathcal{O}\left(n_e^{-1/2}\right)$ holds for a large class of estimators under conditions on the maximum degree. An example is lasso. Under fixed sparsity, bounded regression matrix, and regularities in Negahban et al. (2012), $\zeta_m = 1/2$, $\mathcal{C}_{\mathcal{M}} = \log(p)$, where p is the dimension of the regression matrix. To attain $\mathcal{N}_n^{1/2}C_{\mathcal{M}}^{1/2}C_{\mathcal{E}}^{1/2}/n_e^{\zeta_m+\zeta_e} = \mathcal{O}\left(n_e^{-1/2}\right)$, we only need that ζ_e for the propensity score is such that $\mathcal{N}_n^{1/2}\mathcal{C}_{\mathcal{E}}^{1/2}\log^{1/2}(p)/n_e^{\zeta_e} = \mathcal{O}(1)$.

B.3 Welfare with spillovers on non-compliance

Consider the setting where spillovers also occur over individuals' compliance. Namely, let $D_i \in \{0,1\}$ denote the assigned treatment and $S_i \in \{0,1\}$ denote the selected treatment

[3]The mean-squared error is bounded by $\mathcal{O}(1/\delta_n^2)$. Therefore, a smaller δ_n leads to *slower* convergence rates, captured in the right-hand side of (B.2).

[4]For those units i with a finite number of observations in their partition k, $\sum_{v=1}^n R_v\phi_v^m(i) = \mathcal{O}(1)$, and $\mathcal{O}\left(\mathbb{E}\Big[\Big(1+\sum_{v=1}^n R_v\phi_v^m(i)\Big)^{-2\zeta_m}\Big|A,Z,R_i=1\Big]\right)$ is bounded away from (does not converge to) zero for i.

from individual i. I model non-compliance as follows:

$$Y_i = r\Big(S_i, \sum_{k\in N_i} S_k, Z_i, |N_i|, \varepsilon_i\Big), \quad S_i = h_\theta\Big(D_i, \sum_{k\in N_i} D_k, Z_i, |N_i|, \nu_i\Big). \tag{B.3}$$

I let ν_i be exogenous unobservables, independent from ε_i (see Proposition B.3), and $(r(\cdot), \theta)$ unknown, with θ denoting the set of parameters indexing h. Similarly to what discussed in Section 2, let $W(\pi) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[Y_i\middle|A, Z, \Big\{D_i = \pi(X_i)\Big\}_{i=1}^n\right]$, the effect under π, now also mediated through non-compliance. I discuss identification below.

Proposition B.3. *Let Equation (B.3) hold with* $\varepsilon_i \perp \Big((\nu_j)_{j=1}^n, (\varepsilon_{D_j})_{j=1}^n\Big)\Big|A, Z,$ $\nu_i\Big|A, Z, (\varepsilon_{D_j})_{j=1}^n \sim_{i.i.d.} \mathcal{P}_\nu$. *Let* $P_\theta(S_i = 1|\cdot)$ *denotes the conditional probability of selection into treatment indexed by the parameters* θ. *For each* $i \in \{1, \cdots, n\}$,

$$\mathbb{E}\left[Y_i\middle|A, Z, \Big\{D_i = \pi(X_i)\Big\}_{i=1}^n\right] = \sum_{d\in\{0,1\}, s\in\{0,\cdots,|N_i|\}} \mathbb{E}\Big[Y_i\Big|Z_i, |N_i|, S_i = d, \sum_{k\in N_i} S_k = s\Big] \times H_i(d, s, \pi),$$

$$H_i(d, s, \pi) = P_\theta\Big(S_i = d\Big|Z_i, |N_i|, V_i(\pi)\Big) \sum_{u_1,\cdots,u_l:\sum_v u_v = s} \prod_{k=1}^{|N_i|} P_\theta\Big(S_{N_i^{(k)}} = u_k\Big|Z_{N_i^{(k)}}, |N_{N_i^{(k)}}|, V_{N_i^{(k)}}(\pi)\Big),$$

where $V_i(\pi) = \Big\{D_i = \pi(X_i), \sum_{k\in N_i} D_k = \sum_{k\in N_i} \pi(X_k), Z_i, Z_{k\in N_i}\Big\}$.

See Appendix D.4.1 for the proof. Proposition B.3 is an identification result. The welfare effect of an incentive π depends on conditional means and $H_i(\cdot)$. Here $H_i(\cdot)$ denotes the conditional probability of selecting into treatment, conditional on the individual and neighbors' incentives. Its expression only depends on the individual probability of selected treatments $P_\theta(S_i = 1|\cdot)$, conditional on individual's and neighbors' treatment assignments. Interestingly, $H_i(\cdot)$ also depends on the treatment assigned to the second-degree neighbors; therefore, information from second-degree neighbors is required for identification.[5]

B.4 Different and unknown target population

This subsection briefly presents a setting where experiment participants are drawn from a *different* population from the target population, and the target population is *unknown*. The target population is characterized by an adjacency matrix and covariates (A', Z') as in Section 4.3. Suppose we cannot reweight as in Section 4.3. Because we sample from a different and unknown target population, we can only hope to control the average regret $\sup_{\pi\in\Pi_n} \mathbb{E}[W_{A',Z'}(\pi)] - \mathbb{E}[W_{A',Z'}(\hat{\pi}_{m^c,e})]$, where the expectation is also with respect to (A', Z'). This is a *weaker* notion of regret compared to the one studied in Theorem 3.1. It follows

$$\frac{1}{2}\mathbb{E}\Big[\sup_{\pi\in\Pi_n} \mathbb{E}[W_{A',Z'}(\pi)] - W_{A',Z'}(\hat{\pi}_{m^c,e})\Big] \le \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|W_n(\pi, m^c, e) - \mathbb{E}[W_{A',Z'}(\pi)]\Big|\Big]. \tag{B.4}$$

[5] Literature on non compliance includes Kang and Imbens (2016), Vazquez-Bare (2020). These references do not study welfare maximization. This motivates a different identification strategy here.

The results of this paper can be extended to bound Equation (B.4) as follows. Define Λ_n as an *arbitrary* random variable independent of (A', Z'), and similarly Λ_n', independent of $(A, Z, W_n(\pi)), \pi \in \Pi$. Suppose $(A, Z, W_n(\pi), \Lambda_n) \perp (A', Z', \Lambda_n'), \pi \in \Pi$. For any Λ_n, and Λ_n'[6]

$$\begin{aligned}(\text{B.4}) \leq &\mathbb{E}\Big[\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|W_n(\pi, m^c, e) - \mathbb{E}[W_n(\pi, m^c, e)|\Lambda_n]\Big||\Lambda_n\Big]\Big] \\ &+ \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\mathbb{E}[W_n(\pi, m^c, e)|\Lambda_n] - \mathbb{E}[W_{A',Z'}(\pi)|\Lambda_n']\Big|\Big].\end{aligned} \tag{B.5}$$

We want to choose Λ_n (unobserved to researchers), such that $\Big(Y_i, D_i, Z_i, A_{i,1}, \cdots, A_{i,n}\Big)\Big|\Lambda_n$ exhibit local dependence between units. We can then use the proof technique in Section 3.4 to bound the first component in (B.5), while controlling for the largest degree of dependence given Λ_n. The second component depends on distributional restrictions on (A', Z').

If *no* distributional assumption on (A', Z') is imposed, we can let $\Lambda_n = (A, Z)$. In this case, the second component in the right-hand side of Equation (B.5) is non-zero and captures the additional *cost* when sampled units are *not* drawn from the target population. If instead we are willing to impose distributional restrictions on (A, Z, A', Z'), it is possible to control both components for different choices of Λ_n.[7]

Appendix C A numerical study

Next, I present a small numerical study. I simulate data as follows:

$$Y_i = \frac{1}{\max 1, |N_i|}\Big(X_i\beta_1 + X_i\beta_2 D_i + \mu\Big)\sum_{k\in N_i} D_k + X_i\beta_3 D_i + \varepsilon_i, \quad \varepsilon_i = \frac{\eta_i + \sum_{k\in N_i}\eta_k}{\sqrt{2(|N_i|+1)}}, \tag{C.1}$$

with $\eta_i \sim_{i.i.d.} \mathcal{N}(0,1)$. I simulate covariates as $X_i \in [-1,1]^4$, with each entry drawn independently and uniformly between $[-1,1]$. I draw $\beta_3 \in \{-1.5, 1.5\}$ with equal probabilities. I consider five versions of NEWM. The first two only use a (correctly specified) conditional mean function, without propensity score adjustment with and without approximate network cross-fitting where I choose ten folds (Algorithm 3). The latter three use doubly robust estimators, controlling for either the number of treated neighbors (NEWM_dr1) or for a binned function of those (see Remark 1), with and without approximate cross-fitting.

[6] The expression below follows from the triangular inequality and the fact that under independence of (A', Z') with respect to in-sample observations $W_n(\pi)$, $\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\mathbb{E}[W_n(\pi, m^c, e)|\Lambda_n] - \mathbb{E}[W_{A',Z'}(\pi)]\Big|\Big] = \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\mathbb{E}_{\Lambda_n'}[\mathbb{E}[W_n(\pi, m^c, e) - W_{A',Z'}(\pi)|\Lambda_n, \Lambda_n']]\Big|\Big] \leq \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\mathbb{E}[W_n(\pi, m^c, e)|\Lambda_n] - \mathbb{E}[W_{A',Z'}(\pi)|\Lambda_n']\Big|\Big]$, where $\mathbb{E}_{\Lambda_n'}$ is the conditional expectation that integrates with respect to Λ_n' only.

[7] An example are adjacency matrices generated by conditional dependency graphs Λ_n, Λ_n' (e.g., Kojevnikov et al., 2021), under exchangeability of such graphs. Specifically, the proof technique of this paper directly applies to the first component in Equation (B.5) once we choose Λ_n to be a local dependency graph. The second component equals zero (and therefore there is no penalty for drawing target and sample units from different populations in terms of average regret) if $\mathbb{E}[W_n(\pi, m^c, e)|\Lambda_n] = \mathbb{E}[W_{A,Z}(\pi)|\Lambda_n] = \mathbb{E}[W_{A',Z'}(\pi)|\Lambda_n']$.

I compare the method to two competing methods that ignore network effects from Kitagawa and Tetenov (2018); Athey and Wager (2021). Each method uses a policy function of the form $\pi(X_i) = 1\Big\{X_{i,1}\phi_1 + X_{i,2}\phi_2 + \phi_3 \geq 0\Big\}$, estimated via MILP.

First, I consider a geometric network formation of the form $A_{i,j} = 1\Big\{|X_{i,2} - X_{j,2}|/2 + |X_{i,4} - X_{j,4}|/2 \leq r_n\Big\}$ where $r_n = \sqrt{4/2.75n}$ similarly to simulations in Leung (2020). In the second set of simulations, I generate Barabasi-Albert networks.[8] I estimate the methods over 200 data sets with $n_e = n$, and I evaluate the performance of each estimate out-of-sample over 1000 networks, drawn from the same distribution. Results are in Table 1. For n sufficiently large ($n = 200$), the five specifications of NEWM yield comparable results. NEWM outperforms methods that ignore spillover effects *uniformly* across all specifications.

Table 1: Out-of-sample median *welfare* over 200 replications. DR is the method in Athey and Wager (2021) with estimated balancing score and EWM PS is the method in Kitagawa and Tetenov (2018) with known balancing score. NEWM_out1 is NEWM with a correctly specified outcome model, and NEWM_out2 its equivalent with approximate network cross-fitting. NEWM_dr1 is the doubly robust equivalent controlling for the number of treated neighbors, and NEWM_dr2, NEWM_dr3 control for a binned version of the number of treated neighbors as in Remark 1, with and without approximate network cross-fitting respectively. GE denotes the geometric network, and AB the Albert-Barabasi network.

Welfare	$n = 50$		$n = 70$		$n = 100$		$n = 150$		$n = 200$	
	GE	AB	GE	AB	GE	AB	GE	AB	GE	AB
DR	1.51	0.94	1.50	1.08	1.42	1.05	1.53	0.95	1.41	0.95
EWM PS	1.21	0.93	1.23	0.92	1.32	0.93	1.38	0.90	1.29	0.95
NEWM_out1	1.74	1.31	1.87	1.38	1.93	1.37	1.91	1.40	2.00	1.39
NEWM_out2	1.77	1.34	1.87	1.41	1.91	1.37	1.95	1.38	1.98	1.39
NEWM_dr1	1.78	1.22	1.89	1.33	1.89	1.37	1.94	1.28	1.95	1.33
NEWM_dr2	1.69	1.21	1.83	1.36	1.84	1.33	1.82	1.31	1.94	1.38
NEWM_dr3	1.45	1.15	1.75	1.25	1.79	1.28	1.81	1.28	1.88	1.35

Appendix D Derivations

D.1 Notation

Definition D.1 (Proper Cover)**.** Given an adjacency matrix $A \in \mathcal{A}_n$, with n rows and columns, a family $\mathcal{C}_n = \{\mathcal{C}_n(g)\}$ of disjoint subsets $\mathcal{C}_n(1), \mathcal{C}_n(2), \cdots$ of $\{1, \cdots, n\}$ is a proper

[8] I first draw $n/5$ edges uniformly according to Erdős-Rényi graph with probabilities $p = 10/n$, and second, I draw sequentially connections of the new nodes to the existing ones with probability equal to the number of connection of each pre-existing node divided by the overall number of connections.

cover of A if $\cup_g \mathcal{C}_n(g) = \{1, \cdots, n\}$ and $\mathcal{C}_n(g) \subseteq \{1, \cdots, n\}$ consists of units such that for any pair of elements $\{i, k \in \mathcal{C}_n(g), k \neq i\}$, $A_{i,k} = 0$. $\square$

Definition D.2 (Chromatic number)**.** The chromatic number $\chi_n(A)$, denotes the size of the smallest proper cover of A. $\square$

Definition D.3. For a given matrix $A \in \mathcal{A}_n$, I define $A^2 \in \mathcal{A}_n$ the adjacency matrix such that $A_{i,j} = 1$ if (i, j) are either neighbors or they share at least a common neighbor. Similarly $A^M(A)$ is the adjacency matrix obtained after connecting units sharing common neighbors up to M^{th} degree; $N_{i,M}$ is the set of neighbors of individual i for an adjacency matrix A^M. $\square$

The proper cover of A_n^2 is defined as $\mathcal{C}_n^2 = \{\mathcal{C}_n^2(g)\}_{g=1}^{\chi(A^2)}$ with chromatic number $\chi(A_n^2)$. Similarly $\mathcal{C}_n^M = \{\mathcal{C}_n^M(g)\}_{g=1}^{\chi(A^M)}$ with chromatic number $\chi_n(A_n^M)$ is the proper cover of A_n^M. For a given set $\mathcal{C}_n^M(g)$, I denote $|\mathcal{C}_n^M(g)|$ the number of elements in such a set.

I will refer to $\chi(A)$ as $\chi_n(A_n)$ whenever clear from the context. Let

$$e_i^c(\pi) = e^c\Big(\pi(X_i), T_i(\pi), Z_{k\in N_i}, R_{k\in N_i}, Z_i, |N_i|\Big), \quad m_i^c(\pi) = m^c\Big(\pi(X_i), T_i(\pi), Z_i, |N_i|\Big),$$

for given functions e^c, m^c, and $I_i(\pi) = 1\{T_i(\pi) = T_i, \pi(X_i) = D_i\}$, similarly to Equation (9). In the presence of estimation error, define $\hat{e}_i(\pi), \hat{m}_i(\pi)$ their corresponding estimators.

Following Devroye et al. (2013)'s notation, for $x_1^n = (x_1, ..., x_n)$ being arbitrary points in $\mathcal{X}^n$, for a function class $\mathcal{F}$, with $f \in \mathcal{F}$, $f : \mathcal{X} \mapsto \mathbb{R}$, let $\mathcal{F}(x_1^n) = \{f(x_1), ..., f(x_n) : f \in \mathcal{F}\}$.

Definition D.4. For a class of functions $\mathcal{F}$, with $f : \mathcal{X} \mapsto \mathbb{R}$, $\forall f \in \mathcal{F}$ and n data points $x_1, ..., x_n \in \mathcal{X}$ define the l_q-covering number $\mathcal{M}_q\Big(\eta, \mathcal{F}(x_1^n)\Big)$ to be the cardinality of the smallest cover $\{s_1, ..., s_N\}$, with $s_j \in \mathbb{R}^n$, such that for each $f \in \mathcal{F}$, there exist an $s_j \in \{s_1, ..., s_N\}$ such that $(\frac{1}{n}\sum_{i=1}^n |f(x_i) - s_j^{(i)}|^q)^{1/q} < \eta$. For $\bar{F}$ the envelope of $\mathcal{F}$, define the Dudley's integral as $\int_0^{2\bar{F}} \sqrt{\log\Big(\mathcal{M}_1(\eta, \mathcal{F}(x_1^n))\Big)} d\eta$. $\square$

For random variables $X = (X_1, ..., X_n)$, denote $\mathbb{E}_X[.]$ the expectation with respect to X, conditional on the other variables inside the expectation operator.

Definition D.5. Let $X_1, ..., X_n$ be arbitrary random variables. Let $\sigma = \{\sigma_i\}_{i=1}^n$ be *i.i.d* Rademacher random variables ($P(\sigma_i = -1) = P(\sigma_i = 1) = 1/2$), independent of $X_1, ..., X_n$. The empirical Rademacher complexity is $\mathcal{R}_n(\mathcal{F}) = \mathbb{E}_\sigma\Big[\sup_{f\in\mathcal{F}} |\frac{1}{n}\sum_{i=1}^n \sigma_i f(X_i)|\Big| X_1, ..., X_n\Big]$.

D.2 Theorems

I discuss the theorems first. Appendix D.3 presents the lemmas used for these theorems.

The first theorem controls the supremum of the empirical process of interest with respect to $\Pi \supseteq \Pi_n$ as in Assumption 2.5. Theorem D.1 imposes the same assumptions as Theorem 3.1, except that unobservables can be locally dependent up to the M^{th} degree.

Theorem D.1. *Let Assumptions 2.1, 2.2, 2.3 (A), 2.3 (C), 2.3 (D), 2.5, 3.1, 4.1 (A) hold. Consider functions $m^c(\cdot), e^c(\cdot)$ such that for all $d \in \{0,1\}, t \in \mathcal{T}_n$ $m^c(d,t,Z_i,|N_i|) \in [-\Gamma,\Gamma]$, for a finite constant Γ, and $e^c(d,t,Z_{k\in N_i},R_{k\in N_i},Z_i,|N_i|) \in (\gamma\delta_n, 1-\gamma\delta_n)$ almost surely. Suppose that either (or both) (i) $e^c = e$, or (ii) also Assumption 2.3 (B) hold and $m^c = m$. Then for any $n \geq 1, M \geq 2$, and a universal constant $\bar{C} < \infty$*

$$\mathbb{E}\Big[\sup_{\pi\in\Pi}|W_n(\pi,m^c,e^c) - W_{A,Z}(\pi)|\Big|A,Z\Big] \leq \bar{C}\frac{\Gamma}{\gamma\delta_n}\sqrt{\frac{M\mathcal{N}_n^{M+1}\log(\mathcal{N}_n)\mathrm{VC}(\Pi)}{n_e}}. \tag{D.1}$$

Proof of Theorem D.1. I organize the proof as follows. First, I derive a symmetrization argument to bound the supremum of the empirical process in Equation (D.1) with the Rademacher complexity of direct and spillover effects. Second, I bound the Rademacher complexity using Lemmas D.7, D.8. Section 3.4 provides a proof sketch. Define

$$Q_i(\pi,A,Z) = R_i\left[\frac{I_i(\pi)}{e_i^c(\pi)}\Big(Y_i - m_i^c(\pi)\Big) + m_i^c(\pi)\right],$$

where I suppressed the dependence with e^c, m^c. Define $\mathcal{Q}_n(\pi,A,Z)$ the distribution such that $\Big(Q_i(\pi,A,Z)\Big)_{i=1}^n\Big|A,Z \sim \mathcal{Q}_n(\pi,A,Z)$. Define $(\sigma_i)_{i=1}^n$ *i.i.d.* Rademacher random variables independent of observables and unobservables. Finally, let $\Big(Q_i'(\pi,A,Z)\Big)_{i=1}^n\Big|A,Z \sim \mathcal{Q}_n(\pi,A,Z)$, an independent copy of $\Big(Q_i(\pi,A,Z)\Big)_{i=1}^n$, conditional on (A,Z). Note that $Q_i(\pi,A,Z)$ depends on π through $\Big(\pi(X_i), \sum_{k\in N_i}\pi(X_k)\Big)$ by Assumption 2.1 (since $T_i(\pi)$ is an arbitrary function of $\sum_{k\in N_i}\pi(X_k)$).

Conditional expectation By definition of Q_i',

$$\mathbb{E}[W_n(\pi,e^c,m^c)|A,Z] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[Q_i(\pi,e^c,m^c)|A,Z] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[Q_i'(\pi,e^c,m^c)|A,Z]. \tag{D.2}$$

It follows:

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\pi\in\Pi}|W_n(\pi,m^c,e^c) - W_{A,Z}(\pi)|\Big|A,Z\Big] \\
&= \mathbb{E}\Big[\sup_{\pi\in\Pi}|W_n(\pi,m^c,e^c) - \mathbb{E}[W_n(\pi,m^c,e^c)|A,Z]|\Big|A,Z\Big] && (\because \text{Lemma D.10}) \\
&= \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n\Big[Q_i(\pi,A,Z) - \mathbb{E}[Q_i'(\pi,A,Z)|A,Z]\Big]\Big||A,Z\Big] && (\because \text{Eq. (D.2)}) \\
&= \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n\mathbb{E}_{Q'}\Big[Q_i(\pi,A,Z) - Q_i'(\pi,A,Z)\Big|A,Z\Big]\Big||A,Z\Big] && (\because (Q_i')_{i=1}^n \perp (Q_i)_{i=1}^n|A,Z) \\
&\leq \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n\Big[Q_i(\pi,A,Z) - Q_i'(\pi,A,Z)\Big]\Big||A,Z\Big] && (\because \text{Jensen's inequality}).
\end{aligned} \tag{D.3}$$

The second to last equality takes the expectation with respect to Q' (given Q,A,Z).[9]

[9]These steps follow similarly to those in Boucheron et al. (2013).

Symmetrization and proper cover Recall now Definitions D.1, D.2, D.3. Construct an adjacency matrix A^M with neighbors connected up to the M^{th} degree, with smallest proper cover $\mathcal{C}_n^M = \{\mathcal{C}_n(j)\}_{g=1}^{\chi(A^M)}, \mathcal{C}_n^M(g) \subseteq \{1, \cdots, n\}, \cup_g \mathcal{C}_n^M(g) = \{1, \cdots, n\}$, and chromatic number $\chi(A^M)$. Note that such a cover always exists.[10] By the triangular inequality

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n\Big[Q_i(\pi,A,Z) - Q_i'(\pi,A,Z)\Big]\Big||A,Z\Big] \\
&\leq \sum_{j\in\{1,\cdots,\chi(A^M)\}} \underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\Big[Q_i(\pi,A,Z) - Q_i'(\pi,A,Z)\Big]\Big||A,Z\Big]}_{:=II(g)}.
\end{aligned} \tag{D.4}$$

Observe first that $\mathbb{E}[Q_i(\pi, A, Z) - Q_i'(\pi, A, Z)|A, Z] = 0$ since Q_i, Q_i' have the same distribution. Also, if $R_i = 0$, then $Q_i = 0$. Therefore, by Assumption 2.1, and Assumption 2.2 (i), for a given π, $Q_i(\pi, A, Z)$ is a deterministic function of $R_i\Big(R_{k\in N_i}, \varepsilon_{D_i}, \varepsilon_{D_{k\in N_i}}, Z_{k\in N_i}, Z_i, \varepsilon_i, R^f_{k\in N_i}\Big)$. Also, note that if $R_i = 1$, then $R_k^f = 1$, for $k \in N_i$ almost surely. Therefore, Q_i can be written as a deterministic function of $\Big(R_i, R_{k\in N_i}, \varepsilon_{D_i}, \varepsilon_{D_{k\in N_i}}, Z_{k\in N_i}, Z_i, \varepsilon_i\Big)$ only, where we can drop its dependence with $R^f_{k\in N_i}$. The following holds.

- By Assumption 2.2 (i), ε_{D_i} are *i.i.d.* and exogenous with respect to (A, Z, ε);
- By Assumption 2.2 (iii) and Assumption 2.3 (A) R_i are *i.i.d.* and exogenous;
- Under Assumption 4.1 (A), $\varepsilon_i|A, Z$ are independent for individuals who are not neighbors up to degree $M \geq 2$.

As a result, it directly follows that conditional on A, Z, for any $M \geq 2$,

$$\Big(R_i, R_{k\in N_i}, \varepsilon_{D_i}, \varepsilon_{D_{k\in N_i}}, Z_{k\in N_i}, Z_i, \varepsilon_i\Big) \perp \Big(R_j, R_{k\in N_j}, \varepsilon_{D_j}, \varepsilon_{D_{k\in N_j}}, Z_{k\in N_j}, Z_j, \varepsilon_j\Big)_{j\notin\cup_{k=1}^M N_{i,k}}|A, Z. \tag{D.5}$$

Equation (D.5) implies that $Q_i(\pi, A, Z) \perp (Q_j(\pi, A, Z))_{j\notin\cup_{k=1}^M N_{i,k}}|A, Z$. Since $(Q_i)_{i=1}^n, (Q_i')_{i=1}^n|A, Z$ have the same *joint* distribution and are independent, we also have

$$\Big(Q_i(\pi,A,Z) - Q_i'(\pi,A,Z)\Big) \perp \Big(Q_j(\pi,A,Z) - Q_j'(\pi,A,Z)\Big)_{j\notin\cup_{k=1}^M N_{i,k}}|A, Z. \tag{D.6}$$

Note that $(Q_i)_{i\in\mathcal{C}_n^M(g)} =_d (Q_i')_{i\in\mathcal{C}_n^M(g)}|A, Z$ and are independent (since $\mathcal{C}_n^M$ is deterministic conditional on A). Therefore, for each group $\mathcal{C}_n^M(g)$, by Equation (D.6), for $i \in \mathcal{C}_n^M(g)$

$$\Big(Q_i(\pi,A,Z) - Q_i'(\pi,A,Z)\Big) \perp \Big(Q_j(\pi,A,Z) - Q_j'(\pi,A,Z)\Big)_{j\neq i, j\in\mathcal{C}_n^M(g)}|A, Z.$$

[10] For example, in a fully connected network, the chromatic number is n, where each group only contains one unit, while in a network with no connection, the chromatic number is one. The size of such cover (chromatic number) will affect the bound in the statement of the theorem via the maximum degree.

We can then bound $II(g)$ in Equation (D.4) as follows

$$\begin{aligned}
II(g) &= \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\sigma_i\Big[Q_i(\pi,A,Z)-Q_i'(\pi,A,Z)\Big]\Big||A,Z\Big] \\
&\leq \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\sigma_i Q_i(\pi,A,Z)\Big||A,Z\Big] + \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\sigma_i Q_i'(\pi,A,Z)\Big||A,Z\Big] \\
&= 2\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\sigma_i Q_i(\pi,A,Z)\Big||A,Z\Big].
\end{aligned}$$

The first equality follows from independence of $Q_i - Q_i'|A,Z$ within the subset $\mathcal{C}_n^M(g)$, and the fact that Q_i, Q_i' have the same distribution. The second inequality follows from the triangular inequality and Q_i, Q_i' having the same joint distribution given A, Z.

Bound on the Rademacher complexity The following holds

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\sigma_i Q_i(\pi,A,Z)\Big||A,Z\Big] \leq \mathbb{E}\Big[\underbrace{\mathbb{E}_{Y,\sigma}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\sigma_i R_i\frac{I_i(\pi)}{e_i^c(\pi)}Y_i\Big|\Big]}_{:=i(g)}|A,Z\Big] \\
&+\mathbb{E}\Big[\underbrace{\mathbb{E}_{\sigma}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\sigma_i R_i\frac{I_i(\pi)}{e_i^c(\pi)}m_i^c(\pi)\Big|\Big]|A,Z\Big]}_{:=ii(g)} + \underbrace{\mathbb{E}\Big[\mathbb{E}_{\sigma}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^M(g)}\sigma_i R_i m_i^c(\pi)\Big|\Big]|A,Z\Big]}_{:=iii(g)},
\end{aligned} \tag{D.7}$$

where $\mathbb{E}_{Y,\sigma}[\cdot]$ denotes the conditional expectation with respect to (Y,σ) only, given all other observables and unobservables, and similarly $\mathbb{E}_\sigma[\cdot]$, with respect to σ only. Let $\bar{C} < \infty$ be a universal constant. I invoke Lemma D.8 for each element in Equation (D.7) as follows.

- I invoke Lemma D.8 for $i(g)$ with Y_i in lieu of Ω_i in the statement of Lemma D.8, with third moment bounded by Γ^2 by Assumption 2.3 (D); and $\frac{I_i(\pi)}{e_i^c(\pi)}$ in lieu of $g_i(\cdot)$ in Lemma D.8, with upper bound $U_n = 1/(\gamma\delta_n)$ (U_n as in the statement of Lemma D.8) by Assumption 2.2 (ii). Since we sum over elements $R_i 1\{i \in \mathcal{C}_n^M(g)\} = 1$, by Lemma D.8
$$i(g) \leq \bar{C}\frac{\Gamma}{n_e\gamma\delta_n}\sqrt{\mathrm{VC}(\Pi)\mathcal{N}_n\sum_{i=1}^n R_i 1\{i\in\mathcal{C}_n^M(g)\}\log(\mathcal{N}_n)}.$$

- I invoke Lemma D.8 for $ii(g)$ where we have $\frac{I_i(\pi)}{e_i^c(\pi)}m_i(\pi)$ in lieu of $g_i(\cdot)$ in the statement of Lemma D.8, with constant $U_n = \Gamma/(\gamma\delta_n)$ by Assumption 2.3 (D) and Assumption 2.2 (ii), and $\Omega_i = 1$ in the statement of Lemma D.8. Therefore,
$$ii(g) \leq \bar{C}\frac{\Gamma}{n_e\gamma\delta_n}\sqrt{\mathrm{VC}(\Pi)\mathcal{N}_n\sum_{i=1}^n R_i 1\{i\in\mathcal{C}_n^M(g)\}\log(\mathcal{N}_n)}.$$

- I invoke Lemma D.8 for $iii(g)$ where we have $m_i(\pi)$ in lieu of $g_i(\cdot)$ with constant $U_n = \Gamma$, and $\Omega_i = 1$ in the statement of Lemma D.8. Therefore,

$$iii(g) \leq \bar{C}\frac{\Gamma}{n_e}\sqrt{\mathrm{VC}(\Pi)\mathcal{N}_n \sum_{i=1}^{n} R_i 1\{i \in \mathcal{C}_n^M(g)\} \log(\mathcal{N}_n)}.$$

Summing the terms Collecting the terms together, I obtain

$$\text{(D.4)} \leq \sum_{g\in\{1,\cdots,\chi(A^M)\}} \mathbb{E}\Big[\frac{\Gamma}{n_e\gamma\delta_n}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\mathrm{VC}(\Pi)\sum_{i=1}^{n} R_i 1\{i\in\mathcal{C}_n^M(g)\}}\Big|A,Z\Big],$$

where the expectation is taken with respect to $R = (R_1, \cdots, R_n)$. I write

$$\begin{aligned}
&\sum_{g\in\{1,\cdots,\chi(A^M)\}} \mathbb{E}\Big[\frac{\Gamma}{n_e\gamma\delta_n}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\mathrm{VC}(\Pi)\sum_{i=1}^{n} R_i 1\{i\in\mathcal{C}_n^M(g)\}}\Big|A,Z\Big] \\
&\leq \sum_{g\in\{1,\cdots,\chi(A^M)\}} \frac{\Gamma}{n_e\gamma\delta_n}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\mathrm{VC}(\Pi)\sum_{i=1}^{n} \mathbb{E}[R_i|A,Z]1\{i\in\mathcal{C}_n^M(g)\}} \qquad (\because\ \text{Jensen's inequality}) \\
&= \sum_{g\in\{1,\cdots,\chi(A^M)\}} \frac{\Gamma}{n_e\gamma\delta_n}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\mathrm{VC}(\Pi)n_e|\mathcal{C}_n(g)|/n} \qquad (\because \mathbb{E}[R_i|A,Z] = n_e/n).
\end{aligned} \tag{D.8}$$

We have

$$\begin{aligned}
\text{(D.8)} &\leq \chi(A^M)\frac{\Gamma}{n_e\gamma\delta_n}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\mathrm{VC}(\Pi)n_e\frac{1}{\chi(A^M)}\sum_{g\in\{1,\cdots,\chi(A^M)\}}|\mathcal{C}_n^M(g)|/n} \quad (\because \text{concave } \sqrt{x}) \\
&= \chi(A^M)\frac{\Gamma}{n_e\gamma\delta_n}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\mathrm{VC}(\Pi)n_e\frac{1}{\chi(A^M)}} = \frac{\Gamma}{\gamma\delta_n}\sqrt{\frac{\chi(A^M)\mathcal{N}_n\log(\mathcal{N}_n)\mathrm{VC}(\Pi)}{n_e}}.
\end{aligned} \tag{D.9}$$

In the first inequality in (D.9) I divided and multiplied by $\chi(A^M)$ and used concavity of the square-root function. In the second equality I used the fact that $\{\mathcal{C}_n^M(g)\}$ contain disjoint sets, with $\sum_g |\mathcal{C}_n^M(g)| = n$. By Lemma D.2 $\chi(A^M) \leq M\mathcal{N}_n^M$, completing the proof. □

D.2.1 Theorem 3.1 and Theorem 4.2

I state these two theorems as corollaries of Theorem D.1.

Corollary 1. *Theorem 3.1 holds.*

Proof. Following Kitagawa and Tetenov (2018),

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\pi\in\Pi_n} W_{A,Z}(\pi) - W_{A,Z}(\hat{\pi}_{m^c,e})\Big|A,Z\Big] \\
&= \mathbb{E}\Big[\sup_{\pi\in\Pi_n} W_{A,Z}(\pi) - W_n(\hat{\pi}_{m^c,e}, m^c, e) + W_n(\hat{\pi}_{m^c,e}, m^c, e) - W_{A,Z}(\hat{\pi}_{m^c,e})\Big|A,Z\Big] \\
&\leq \mathbb{E}\Big[\sup_{\pi\in\Pi_n} W_{A,Z}(\pi) - W_n(\pi, m^c, e) + W_n(\hat{\pi}_{m^c,e}, m^c, e) - W_{A,Z}(\hat{\pi}_{m^c,e^c})\Big|A,Z\Big].
\end{aligned} \tag{D.10}$$

We have (D.10) $\leq \mathbb{E}\Big[2\sup_{\pi\in\Pi_n}|W_{A,Z}(\pi) - W_n(\pi, m^c, e)|\Big|A, Z\Big] \leq \mathbb{E}\Big[2\sup_{\pi\in\Pi}|W_{A,Z}(\pi) - W_n(\pi, m^c, e)|\Big|A, Z\Big]$ ($\because \Pi_n \subseteq \Pi$). The proof completes by Theorem D.1, with $M = 2$. □

Corollary 2. *Theorem 4.2 holds.*

Proof. Following the argument of Corollary 1, and using the fact that $\Pi_n \subseteq \Pi$, it follows

$$\mathbb{E}\Big[\sup_{\pi\in\Pi_n} W_{A,Z}(\pi) - W_{A,Z}(\hat{\pi}_{\hat{m},\hat{e}})\Big|A,Z\Big] \leq 2\,\underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}|W_n(\pi,m^c,e^c) - W_{A,Z}(\pi)|\Big|A,Z\Big]}_{(I)}$$

$$+\,2\,\underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}|W_n(\pi,\hat{m},\hat{e}) - W_n(\pi,m^c,e^c)|\Big|A,Z\Big]}_{(II)}.$$

Term (I) is bounded by Theorem D.1. I now study (II). In particular, (II) is equal to

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\frac{I_i(\pi)}{\hat{e}_i(\pi)}\Big(Y_i - \hat{m}_i(\pi)\Big) + \frac{1}{n_e}\sum_{i=1}^n R_i\Big(\hat{m}_i(\pi) - m_i^c(\pi)\Big) - \frac{1}{n_e}\sum_{i=1}^n R_i\frac{I_i(\pi)}{e_i^c(\pi)}\Big(Y_i - m_i^c(\pi)\Big)\Big||A,Z\Big]\\
&\leq \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\frac{I_i(\pi)}{\hat{e}_i(\pi)}\Big(Y_i + m_i^c(\pi) - m_i^c(\pi) - \hat{m}_i(\pi)\Big) - \frac{1}{n_e}\sum_{i=1}^n R_i\frac{I_i(\pi)}{e_i^c(\pi)}\Big(Y_i - m_i^c(\pi)\Big)\Big||A,Z\Big]\\
&\quad + \mathbb{E}\Big[\sup_{\pi\in\Pi}|\frac{1}{n_e}\sum_{i=1}^n R_i\Big(\hat{m}_i(\pi) - m_i^c(\pi)\Big)|\Big|A,Z\Big]\\
&\leq \mathbb{E}\Big[\sup_{\pi\in\Pi}|\frac{1}{n_e}\sum_{i=1}^n R_i\Big(\hat{m}_i(\pi) - m_i^c(\pi)\Big)|\Big|A,Z\Big] + \mathbb{E}\Big[\sup_{\pi\in\Pi}|\frac{1}{n_e}\sum_{i=1}^n R_i(\frac{I_i(\pi)}{e_i^c(\pi)} - \frac{I_i(\pi)}{\hat{e}_i(\pi)})\Big(Y_i - m_i^c(\pi)\Big)|\Big|A,Z\Big]\\
&\quad + \mathbb{E}\Big[\sup_{\pi\in\Pi}|\frac{1}{n_e}\sum_{i=1}^n R_i\frac{I_i(\pi)}{\hat{e}_i(\pi)}\Big(\hat{m}_i(\pi) - m_i^c(\pi)\Big)|\Big|A,Z\Big].
\end{aligned} \tag{D.11}$$

I inspect each term in Equation (D.11). Since $R_i \in \{0,1\}$

$$\mathbb{E}\Big[\sup_{\pi\in\Pi}|\frac{1}{n_e}\sum_{i=1}^n R_i\Big(\hat{m}_i(\pi) - m_i^c(\pi)\Big)|\Big|A,Z\Big] \leq \mathbb{E}\Big[\frac{1}{n_e}\sum_{i=1}^n \sup_{d,s} R_i|\hat{m}(d,s,Z_i,|N_i|) - m^c(d,s,Z_i,|N_i|)|\Big|A,Z\Big].$$

By Cauchy-Schwarz inequality and the triangular inequality

$$\begin{aligned}
&\mathbb{E}\Big[\frac{1}{n_e}\sum_{i=1}^n \sup_{d,s} R_i|\hat{m}(d,s,Z_i,|N_i|) - m^c(d,s,Z_i,|N_i|)|\Big|A,Z\Big]\\
&\leq \sqrt{\frac{1}{n_e}\sum_{i=1}^n \mathbb{E}[R_i^2]}\sqrt{\mathbb{E}\Big[\frac{1}{n_e}\sum_{i=1}^n \sup_{d,s}|\hat{m}(d,s,Z_i,|N_i|) - m^c(d,s,Z_i,|N_i|)|^2\Big|A,Z\Big]}\\
&= \sqrt{\mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^n \sup_{d,s}|\hat{m}(d,s,Z_i,|N_i|) - m^c(d,s,Z_i,|N_i|)|^2\Big|A,Z\Big]} \quad (\because \mathbb{E}[R_i^2] = \mathbb{E}[R_i] = n_e/n).
\end{aligned}$$

For the second term we have (let $e^c_i(d,t) = e^c(d,t,Z_{k\in N_i}, R_{k\in N_i}, |N_i|)$ and similarly for $\hat{e}_i(d,t)$)

$$\mathbb{E}\Big[\sup_{\pi\in\Pi}|\frac{1}{n_e}\sum_{i=1}^n R_i(\frac{I_i(\pi)}{e^c_i(\pi)} - \frac{I_i(\pi)}{\hat{e}_i(\pi)})\Big(Y_i - m^c_i(\pi)\Big)|\Big|A,Z\Big] \leq 2\Gamma'\mathbb{E}\Big[\sup_{\pi\in\Pi}\frac{1}{n_e}\sum_{i=1}^n R_i|(\frac{I_i(\pi)}{e^c_i(\pi)} - \frac{I_i(\pi)}{\hat{e}_i(\pi)})|\Big|A,Z\Big]$$

$$\leq 2\Gamma'\mathbb{E}\Big[\frac{1}{n_e}\sum_{i=1}^n R_i \sup_{d,t}|(\frac{1}{e^c_i(d,t)} - \frac{1}{\hat{e}_i(d,t)})|\Big|A,Z\Big] \leq 2\Gamma'\sqrt{\mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^n \sup_{d,t}|(\frac{1}{e^c_i(d,t)} - \frac{1}{\hat{e}_i(d,t)})|^2\Big|A,Z\Big]}$$

where in the first inequality I used the fact that Y, m^c are uniformly bounded and in the last inequality I used Cauchy-Schwarz. For the third term in (D.11), it follows similarly

$$\mathbb{E}\Big[\sup_{\pi\in\Pi}|\frac{1}{n_e}\sum_{i=1}^n R_i\frac{I_i(\pi)}{\hat{e}_i(\pi)}\Big(\hat{m}_i(\pi) - m^c_i(\pi)\Big)|\Big|A,Z\Big] \leq \frac{1}{\gamma\delta_n}\mathbb{E}\Big[\sup_{\pi\in\Pi}\frac{1}{n_e}\sum_{i=1}^n R_i|\Big(\hat{m}_i(\pi) - m^c_i(\pi)\Big)|\Big|A,Z\Big]$$

$$\leq \frac{1}{\gamma\delta_n}\sqrt{\mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^n \sup_{d,t}|\Big(\hat{m}(d,t,Z_i,|N_i|) - m^c(d,t,Z_i,|N_i|)\Big)|^2\Big|A,Z\Big]}.$$

D.2.2 Proof of Theorem 3.2

The proof constructs an appropriate adjacency matrix, matrix of covariates and distribution of treatments and unobservables to provide the lower bound, taking into account the selection indicators. Recall the definition of $\mathbb{E}_{\mathcal{D}_n(A,Z)}[\cdot]$ in Theorem 3.2. Let $v = \mathrm{VC}(\Pi)$, and recall, under Assumption 2.2 (iii), $R_i \sim_{i.i.d.} \mathrm{Bern}(\alpha), \alpha = n_e/n$. Let $X_i = Z_i$ for expositional convenience.[11] Let $A^* \in \mathcal{A}^o_n$, such that $A^*_{i,j} = 0$ for all $i \neq j$. Let $z_1, \cdots, z_v$ be v points shattered by Π, which, since $\mathcal{X} = \mathbb{R}^d$ and Π has VC dimension v they must exist. Let Z^* such that $\frac{1}{n}\sum_{i=1}^n 1\{Z^*_i = z_j\} = \frac{1}{v}$ for all $j \in \{1, \cdots, v\}$. I write

$$\sup_{A\in\mathcal{A}^o_n, Z\in\mathcal{Z}^n}\ \sup_{\mathcal{D}_n(A,Z)\in\mathcal{P}_n(A,Z)} \frac{\delta_n}{\mathcal{N}_n^{3/2}\log^{1/2}(\mathcal{N}_n)}\mathbb{E}_{\mathcal{D}_n(A,Z)}\Big[\Big(\sup_{\pi\in\Pi} W_{A,Z}(\pi) - W_{A,Z}(\hat{\pi}_n)\Big)\Big|A,Z\Big]$$
$$\geq \sup_{\mathcal{D}_n(A^*,Z^*)\in\mathcal{P}_n(A^*,Z^*)} \frac{\delta_n}{\mathcal{N}_n^{3/2}\log^{1/2}(\mathcal{N}_n)}\mathbb{E}_{\mathcal{D}_n(A^*,Z^*)}\Big[\Big(\sup_{\pi\in\Pi} W_{A^*,Z^*}(\pi) - W_{A^*,Z^*}(\hat{\pi}_n)\Big)\Big|A = A^*, Z = Z^*\Big], \tag{D.12}$$

where, recall that $\delta_n, \mathcal{N}_n$ are also a function of A^*, Z^*.

I will focus on Equation (D.12). I will indicate for $|A^*, Z^*$ the conditioning set $|A = A^*, Z = Z^*$. Because I consider a fully disconnected network, we have $\delta_n = 1$ in Assumption 2.2 (since individuals have no neighbors), and $\mathcal{N}_n = 2$ for adjacency matrix A^*. I follow the proof of Theorem 14.5 in Devroye et al. (2013), and Theorem 2.2 in Kitagawa and Tetenov (2018), while I also condition on (A^*, Z^*), and consider random indicators R_i.

[11] I assume that $X_i = Z_i$ so that I do not have to keep track of both Z_i, X_i through the proof.

Treatment assignments and potential outcomes' distribution Next, I select the distribution for treatment assignments and potential outcomes. Let D_i be a Bernoulli random variable, independent of observables and unobservables with $P(D_i = 1) = 1/2$. Let $\mathbf{b} \in \{0,1\}^v$ be a bit indicator which indexes a distribution $\mathcal{D}_{n,\mathbf{b}}(A^*, Z^*) \in \mathcal{P}_n(A^*, Z^*)$. Namely, I restrict the class of distributions to a finite number of distributions, indexed by $\mathbf{b}$. Denote $Y_i(d) = r(d, 0, Z_i, 0, \varepsilon_i)$, the potential outcome function, where spillovers and number of connections are equal to zero by construction of A^*. Let $P(Y_i(1) = 1/2|Z_i = z_j) = 1/2 + \eta$, $P(Y_i(1) = -1/2|Z_i = z_j) = 1/2 - \eta$ for $\mathbf{b}_j = 1, j \leq v$. If $\mathbf{b}_j = 0$, instead have $P(Y_i(1) = 1/2|Z_i = z_j) = 1/2 - \eta$, $P(Y_i(1) = -1/2|Z_i = z_j) = 1/2 + \eta$, where $\eta \in [0, 1/2]$ and is selected at the end of the proof. Consider $Y_i(0) = 0$ almost surely.

Lower bound via Bayes risk I can therefore write the optimal treatment rule as $\pi^*_{\mathbf{b}}(z_j) = 1\{b_j = 1\}, j \leq v$, which satisfies the finite VC dimension. I have $W_{A^*,Z^*}(\pi^*_{\mathbf{b}}) = \frac{\eta}{v}\sum_{j=1}^v \mathbf{b}_j$ under the distribution $\mathcal{D}_{n,\mathbf{b}}$. Consider $\mathbf{b}$ being a random variable with $\mathbf{b}_j \sim_{i.i.d.} \text{Bern}(1/2)$ and independent of observables and unobservables. Denote $\mathbb{E}_{\mathbf{b}}[\cdot]$ the expectation with respect to $\mathbf{b}$ (conditional on A^*, Z^*). For any data-dependent $\hat{\pi}_n$,[12]

$$
\begin{aligned}
&\sup_{\mathcal{D}_n(A^*,Z^*)\in\mathcal{P}_n} \mathbb{E}_{\mathcal{D}_n(A^*,Z^*)}\Big[W_{A^*,Z^*}(\pi^*_{\mathbf{b}}) - W_{A^*,Z^*}(\hat{\pi}_n)\Big|A^*, Z^*\Big] \\
&\geq \mathbb{E}_{\mathbf{b}}\Big[\mathbb{E}_{\mathcal{D}_{n,\mathbf{b}}(A^*,Z^*)}\Big[W_{A^*,Z^*}(\pi^*_{\mathbf{b}}) - W_{A^*,Z^*}(\hat{\pi}_n)\Big|A^*, Z^*\Big]\Big|A^*, Z^*\Big], \\
&\geq \inf_{\hat{\pi}_n} \eta\frac{1}{v}\sum_{j=1}^v \mathbb{E}_{\mathbf{b}}\Big[\mathbb{E}_{\mathcal{D}_{n,\mathbf{b}}(A^*,Z^*)}\Big[1\{b_j \neq \hat{\pi}_n(z_j)\}\Big|A^*, Z^*\Big]\Big|A^*, Z^*\Big].
\end{aligned}
\tag{D.13}
$$

We can see the minimization in Equation (D.13) as a risk-minimization problem with lower bound provided by the Bayes risk. I construct a Bayes classifier of the form

$$
\hat{\pi}_n(z_j) = 1\left\{P\left(\mathbf{b}_j = 1\big|\Big[(Y_i, D_i, D_{k\in N_i})R_i, R_i\Big]_{i=1}^n, A^*, Z^*\right) \geq 1/2\right\}, j \leq v.
$$

I can then follow the same steps of Kitagawa and Tetenov (2018), Equation (A.12), (A.13), with $k_j^+ = \#\Big\{i : Z_i = z_j, R_i Y_i D_i = 1/2\Big\}, k_j^- = \#\Big\{i : Z_i = z_j, R_i Y_i D_i = -1/2\Big\}$ for the case of this paper, and $Y_i D_i R_i$ in lieu of $Y_i D_i$ in the derivation of Kitagawa and Tetenov (2018). Following (A.12), (A.13), and the equation below (A.13) in Kitagawa and Tetenov (2018)

$$
\begin{aligned}
&\inf_{\hat{\pi}_n} \eta\frac{1}{v}\sum_{j=1}^v \mathbb{E}_{\mathbf{b}}\Big[\mathbb{E}_{\mathcal{D}_{n,\mathbf{b}}(A^*,Z^*)}\Big[1\{b_j \neq \hat{\pi}_n(z_j)\}\Big|A^*, Z^*\Big]\Big|A^*, Z^*\Big] \\
&\geq \frac{\eta}{2v}\sum_{j=1}^v a^{-\mathbb{E}_{\mathbf{b}}\Big[\mathbb{E}_{\mathcal{D}_{n,\mathbf{b}}(A^*,Z^*)}\Big[|\sum_{i:Z_i^*=z_j} 2Y_i D_i R_i|\Big|A^*,Z^*\Big]\Big]}, \quad a = \frac{1+2\eta}{1-2\eta}.
\end{aligned}
$$

[12] See e.g., Appendix A.2 in Kitagawa and Tetenov (2018), Page 8.

Lower bound on the Bayes risk The marginal distribution of $Y_i(1)$ (once we integrate over $\mathbf{b}$), is $P(Y_i(1) = 1/2|Z^*, A^*) = P(Y_i(1) = -1/2|Z^*, A^*) = 1/2$ similarly to Kitagawa and Tetenov (2018). By independence, $P(D_i R_i = 1) = \alpha/2$. We have

$$\mathbb{E}_{\mathbf{b}}\left[\mathbb{E}_{\mathcal{D}_{n,\mathbf{b}}(A^*,Z^*)}\Big[|\sum_{i:Z_i^*=z_j} 2Y_i D_i R_i|\Big|A^*, Z^*\Big]\right] = \mathbb{E}_{\mathbf{b}}\left[\mathbb{E}_{\mathcal{D}_{n,\mathbf{b}}(A^*,Z^*)}\Big[|\sum_{i:Z_i^*=z_j, R_i D_i=1} 2Y_i|\Big|A^*, Z^*\Big]\right]$$
$$= \sum_{k=0}^{n/v}\binom{n/v}{k}(\frac{\alpha}{2})^k(1-\frac{\alpha}{2})^{n/v-k}\mathbb{E}\Big|B(k,\frac{1}{2}) - k/2\Big|, \tag{D.14}$$

where $B(k, 1/2)$ is a binomial random variable with parameters $(k, 1/2)$. Equation (D.14) holds because given $Z = Z^*$, there are n/v many observations with $Z_i^* = z_j, j \leq v$ by construction of Z^*. We can write $\mathbb{E}\Big|B(k,\frac{1}{2}) - k/2\Big| \leq \sqrt{\mathbb{E}\Big(B(k,\frac{1}{2}) - k/2\Big)^2} = \sqrt{\frac{k}{4}}$. It follows

$$\text{(D.14)} \leq \sum_{k=0}^{n/v}\binom{n/v}{k}(\frac{\alpha}{2})^k(1-\frac{\alpha}{2})^{n/v-k}\sqrt{\frac{k}{4}} = \mathbb{E}\sqrt{\frac{B(n/v,\frac{\alpha}{2})}{4}} \leq \sqrt{\frac{\mathbb{E}[B(n/v,\frac{\alpha}{2})]}{4}} = \sqrt{\frac{n\alpha}{v8}}.$$

Following Kitagawa and Tetenov (2018), equation (A.14) and below, with αn in lieu of n in Kitagawa and Tetenov (2018), it follows that the Bayes risk is bounded from below by $\frac{1}{2}\sqrt{\frac{v}{\alpha n}}\exp(-2\sqrt{2})$ for $\alpha n \geq 16v$. Since $n_e = \alpha n, \mathcal{N}_n \leq 2$ for A^*, the proof completes.

D.2.3 Proof of Theorem 3.3

For the sake of brevity, I will be using the following notation

$$\tilde{I}_i(d,t) = 1\Big\{d = D_i, t = T_i\Big\}, \quad \tilde{e}_i(d,t) = e\Big(d, t, Z_{k\in N_i}, R_{k\in N_i}, Z_i, |N_i|\Big), \quad \tilde{m}_i(d,t) \ = m\Big(d, t, Z_i, |N_i|\Big).$$

Also, let $\tilde{\varepsilon}_i = Y_i - m(D_i, T_i, Z_i, |N_i|)$. With an abuse of notation, I will refer to $\hat{e}_i(d,t), \hat{m}_i(d,t)$ as the estimated counterpart of $\tilde{e}_i(d,t), \tilde{m}_i(d,t)$ from Algorithm 2, with arguments (d,t). Let $I_i(\pi), e_i(\pi), m_i(\pi)$ be defined as at the beginning of Section 3.1, and $\hat{e}_i(\pi), \hat{m}_i(\pi)$ be defined as in Algorithm 2 (Equation (26)), as a function of the treatment assignment rule π (therefore $\hat{e}_i(\pi) := \hat{e}_i(\pi(X_i), T_i(\pi))$ and similarly for $\hat{m}_i(\pi)$). Recall the definitions of K^*, F_k^j in Algorithm 2: K^* denotes the number of partitions obtained under Algorithm 2, where we have $k \in \{1, \cdots, K^*\}$ many partitions. Within each partition, we have $j \in \{1, \cdots, J\}$ folds F_k^j. For each $k \in \{1, \cdots, K^*\}$, $\cup_{j=1}^J F_k^j$ never contains two units that are either neighbors or share a common neighbor. Let $R = (R_1, \cdots, R_n)$.

The argument I present in the current proof applies to any K^* obtained from Algorithm 2, and any configurations of folds $(F_k^j)_{j=1}^J, k \in \{1, \cdots, K^*\}$ obtained from Algorithm 2, including settings with folds F_k^j with one or few units.[13]

[13]Specifically, note that Algorithm 2 estimates $\hat{m}^{(i)}, 1/\hat{e}^{(i)}$ as zero functions for those units i, assigned to

Preliminary decomposition Following the same argument of Corollary 1, since $\Pi_n \subseteq \Pi$,

$$\mathbb{E}\Big[\sup_{\pi\in\Pi_n} W_{A,Z}(\pi) - W_{A,Z}(\hat{\pi}_{\hat{m},\hat{e}})\Big|A,Z\Big] \leq 2\underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}|W_n(\pi,m,e) - W_{A,Z}(\pi)|\Big|A,Z\Big]}_{(I)} + 2\underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}|W_n(\pi,\hat{m},\hat{e}) - W_n(\pi,m,e)|\Big|A,Z\Big]}_{(II)}.$$

Term (I) is bounded by Theorem D.1. I now study (II).

$$\begin{aligned}(II) &= \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)}(m_i(\pi)-\hat{m}_i(\pi)) + \tilde{\varepsilon}_i\frac{I_i(\pi)}{\hat{e}_i(\pi)} + \hat{m}_i(\pi) - m_i(\pi)\Big)\Big||A,Z\Big] \\ &= \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)(m_i(\pi)-\hat{m}_i(\pi)) + R_i\tilde{\varepsilon}_i\frac{I_i(\pi)}{\hat{e}_i(\pi)} + R_i\Big(\frac{I_i(\pi)}{e_i(\pi)} - 1\Big)(\hat{m}_i(\pi)-m_i(\pi))\Big||A,Z\Big].\end{aligned}$$

The last equality follows after adding and subctracting $R_i\frac{I_i(\pi)}{e_i(\pi)}(m_i(\pi)-\hat{m}_i(\pi))$. It follows

$$\begin{aligned}(II) \leq &\underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)(m_i(\pi)-\hat{m}_i(\pi))\Big||A,Z\Big]}_{(i)} + \underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\tilde{\varepsilon}_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big||A,Z\Big]}_{(ii)} \\ &+ \underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\tilde{\varepsilon}_i\frac{I_i(\pi)}{e_i(\pi)}\Big||A,Z\Big]}_{(iii)} + \underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\Big(\frac{I_i(\pi)}{e_i(\pi)} - 1\Big)(\hat{m}_i(\pi)-m_i(\pi))\Big||A,Z\Big]}_{(iv)}.\end{aligned} \tag{D.15}$$

We bound each component $(i), (ii), (iii), (iv)$ separately below.

Bounding (i) Consider (i) first. We have

$$\begin{aligned}(i) &= \mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i=1}^n R_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)R_i(m_i(\pi)-\hat{m}_i(\pi))\Big||A,Z\Big] \quad (\because R_i \in \{0,1\}) \\ &\leq \sqrt{\frac{1}{n_e}\mathbb{E}\Big[\sum_{i=1}^n R_i\sup_{d,t}\Big(\frac{1}{\tilde{e}_i(d,t)} - \frac{1}{\hat{e}_i(d,t)}\Big)^2\Big|A,Z\Big]}\sqrt{\frac{1}{n_e}\mathbb{E}\Big[\sum_{i=1}^n R_i\sup_{d,t}\Big(\tilde{m}_i(d,t) - \hat{m}_i(d,t)\Big)^2\Big|A,Z\Big]} \\ &= \sqrt{\mathbb{E}[R_i/n_e]\mathbb{E}\Big[\sum_{i=1}^n\sup_{d,t}\Big(\frac{1}{\tilde{e}_i(d,t)} - \frac{1}{\hat{e}_i(d,t)}\Big)^2\Big|R_i=1,A,Z\Big]} \times \qquad (\because \text{Defn of conditional expectation}) \\ &\quad\times\sqrt{\mathbb{E}[R_i/n_e]\mathbb{E}\Big[\sum_{i=1}^n\sup_{d,t}\Big(\tilde{m}_i(d,t) - \hat{m}_i(d,t)\Big)^2\Big|R_i=1,A,Z\Big]} = \sqrt{\mathcal{R}_n(A,Z)\times\mathcal{B}_n(A,Z)}.\end{aligned} \tag{D.16}$$

groups $k \in \{1,\cdots,K^*\}$ with few (a finite) number of units. The estimation error for such units contributes directly to the average error in Equation (D.16), without affecting the derivation of Theorem 3.3. Appendix B.2 show how to control the estimation error in Equation (D.16).

Summands in (ii) **and** (iii)**,** (iv) Next, I show that each summand in $(ii), (iii), (iv)$ has a zero conditional expectation, given R, A, Z, for any $\hat{e}^{(i)}, \hat{m}^{(i)}$ in Algorithm 2.

(ii) I start from summands in (ii). I write the expectation of each summand as

$$\begin{aligned}
&\mathbb{E}\Big[R_i\tilde{\varepsilon}_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big|R, A, Z\Big] \\
&= \mathbb{E}\Big[R_i\Big(r(\pi(X_i), T_i(\pi), Z_i, |N_i|, \varepsilon_i) - m_i(\pi)\Big)\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big|R, Z, A\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[R_i\Big(r(\pi(X_i), T_i(\pi), Z_i, |N_i|, \varepsilon_i) - m_i(\pi)\Big)\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big|\hat{e}_i(\pi), R, Z, A\Big]\Big|R, Z, A\Big] \\
&= R_i\underbrace{\mathbb{E}\Big[\Big(r(\pi(X_i), T_i(\pi), Z_i, |N_i|, \varepsilon_i) - m_i(\pi)\Big)|A, Z, R\Big]}_{=0}\mathbb{E}\Big[\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big|R, Z, A\Big] \\
&(\because \text{ Alg 2 and Assumptions 2.2}(i), 2.3) \qquad = 0.
\end{aligned} \tag{D.17}$$

The last equality follows from the fact that $T_i(\pi)$ (in Equation (5)) is a deterministic function of (A, Z), ε_i is independent of $\hat{e}_i(\pi)$ given (R, Z, A) by Algorithm 2, and ε_i is conditionally independent of $(D_i, R_i)_{i=1}^n$ given A, Z, by Assumption 2.2 (i), 2.3 (A).

(iii) For (iii), $\mathbb{E}[R_i\tilde{\varepsilon}_i I_i(\pi)/e_i(\pi)|R, A, Z] = 0$ directly by Assumptions 2.2 (i), 2.3 (A).

(iv) For summands in (iv), we have:

$$\begin{aligned}
&\mathbb{E}\Big[R_i\Big(\frac{I_i(\pi)}{e_i(\pi)} - 1\Big)(\hat{m}_i(\pi) - m_i(\pi))\Big|R, A, Z\Big] \\
&= R_i\underbrace{\mathbb{E}\Big[\Big(\frac{I_i(\pi)}{e_i(\pi)} - 1\Big)\Big|R, A, Z\Big]}_{=0}\mathbb{E}\Big[(\hat{m}_i(\pi) - m_i(\pi))\Big|R, A, Z\Big] = 0.
\end{aligned} \tag{D.18}$$

The first equality follows because $\hat{m}_i(\pi)$ is independent of $(D_i, D_{k\in N_i})$ conditional on (R, A, Z) by Algorithm 2 and Assumption 2.2 (i).

Bounds for (ii) Using the triangular inequality and the law of iterated expectations, I write (letting $\hat{e}_i(\cdot)$ be the estimated propensity score function for i)

$$(ii) \leq \mathbb{E}\Big[\sum_{k=1}^{K^*}\sum_{j=1}^{J}\underbrace{\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in F_k^j}R_i\tilde{\varepsilon}_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big|\Big|\hat{e}_{i\in F_k^j}(\cdot), R, A, Z\Big]}_{:=(M_k^j)}\Big|A, Z\Big], \tag{D.19}$$

where here we also condition on R and the estimated functions $\hat{e}_i$ for units in the fold $i \in F_k^j$. Next, we bound each component (M_k^j) in (D.19). We make the following observations.

(1) $(F_k^j)_{j=1}^J, K^*$ are deterministic functions of (R, A) by construction of Algorithm 2.

(2) For each $i \in F_k^j$, $\mathbb{E}\Big[R_i\tilde{\varepsilon}_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big|A, R, Z, \hat{e}_{i\in F_k^j}(\cdot)\Big] = 0$ by (D.17) and independence of $\hat{e}_{i\in F_k^j}(\cdot)$ with $\tilde{\varepsilon}_i$ (independence follows from Alg 2 and Assumptions 2.2 (i), 2.3 (A)).[14]

(3) Conditional on $(\hat{e}_{i\in F_k^j}(\cdot), R, A, Z)$, we have that $\Big\{R_i\tilde{\varepsilon}_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)}(\cdot) - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big\}$ are mutually independent among units in the same fold ($i \in F_k^j$), by 2.2 (i), 2.3 (A) and Alg 2.

Therefore, by (2), and (3) above I can invoke standard symmetrization arguments for centered independent random variables (see Lemma 6.4.2 in Vershynin, 2018) to bound

$$(M_k^j) \leq 2\mathbb{E}\Big[\mathbb{E}_{\tilde{\varepsilon},\sigma}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in F_k^j}\sigma_i R_i\tilde{\varepsilon}_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big|\Big]\big|\hat{e}_{i\in F_k^j}(\cdot), R, A, Z\Big] \tag{D.20}$$

for $(\sigma_1, \cdots, \sigma_n)$ be *i.i.d.* exogenous Radamacher random variables (recall that $\mathbb{E}_{\tilde{\varepsilon},\sigma}[\cdot]$ indicates that the inner expectation is conditional on everything else except $\sigma, \tilde{\varepsilon}$).

I can now directly use Lemma D.8 to bound the right-hand-side of (D.20). Namely, I invoke Lemma D.8 where Ω_i in the statement of Lemma D.8 is $\tilde{\varepsilon}_i$ in Equation (D.20), $g_i(\cdot)$ in Lemma D.8 is $\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)$ in Equation (D.20); U_n in the statement of Lemma D.8 is $\frac{2}{\gamma\delta_n}$ in (D.20). Therefore, by Lemma D.8, for a universal constant $\bar{C} < \infty$

$$\mathbb{E}_{\tilde{\varepsilon},\sigma}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in F_k^j}\sigma_i R_i\tilde{\varepsilon}_i\Big(\frac{I_i(\pi)}{\hat{e}_i(\pi)} - \frac{I_i(\pi)}{e_i(\pi)}\Big)\Big|\Big] \leq \frac{\bar{C}\Gamma}{n_e}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\sum_{i=1}^n R_i 1\{i \in F_k^j\}\mathrm{VC}(\Pi)}.$$

It follows

$$\begin{aligned}
\sum_{j=1}^J \mathbb{E}\Big[\sum_k^{K^*}(M_k^j)\Big|A, Z\Big] &\leq J\mathbb{E}\Big[K^*\frac{\bar{C}\Gamma}{n_e}\sqrt{\frac{\sum_{j=1}^J\sum_{k=1}^{K^*}\mathcal{N}_n\log(\mathcal{N}_n)\sum_{i=1}^n R_i 1\{i\in F_k^j\}\mathrm{VC}(\Pi)}{JK^*}}\Big|A, Z\Big] \\
&\qquad (\because \text{concavity of } \sqrt{x}) \\
&\leq \mathbb{E}\Big[\sqrt{JK^*}\frac{\bar{C}\Gamma}{n_e}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\sum_{i=1}^n R_i\mathrm{VC}(\Pi)}\Big|A, Z\Big] \quad (\because \cup_{k=1,j=1}^{K^*,J}F_k^j \subseteq \{1, \cdots, n\}) \\
&\leq \mathbb{E}\Big[\sqrt{J\chi(A^2)}\frac{\bar{C}\Gamma}{n_e}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\sum_{i=1}^n R_i\mathrm{VC}(\Pi)}\Big|A, Z\Big] \quad (\because K^* \leq \chi(A^2) \text{ by Lem } D.9) \\
&\leq \sqrt{J\chi(A^2)}\frac{\bar{C}\Gamma}{n_e}\sqrt{\mathcal{N}_n\log(\mathcal{N}_n)\sum_{i=1}^n\mathbb{E}[R_i]\mathrm{VC}(\Pi)} \quad (\because \text{Jensen's inequality}).
\end{aligned} \tag{D.21}$$

[14] Independence follows from the fact that $\cup_{j=1}^J F_k^j$ does not contain two sampled individuals that are either neighbors or share a common neighbor. Therefore, we never use information from $(D_i, D_{k\in N_i})$ to estimate $\hat{e}_i(\cdot)$ for all $i : R_i = 1$. Also, note that the argument holds if, for estimating the propensity score for i, we also use information from the neighbors of the units in $\cup_{j=1}^J F_k^j \setminus F_k^{j(i)}$ which have *not* been sampled, where $F_k^{j(i)}$ denotes the fold containing unit i. These units (i.e., non-sampled neighbors of elements in $\cup_{j=1}^J F_k^j \setminus F_k^{j(i)}$) cannot be neighbors of i (for i such that $R_i = 1$) since $\cup_{j=1}^J F_k^j$ does not contain sampled units with a common neighbor.

By Assumption 2.2 (iii) (D.21) $\leq \sqrt{J\chi(A^2)}\bar{C}\Gamma\sqrt{\frac{\mathcal{N}_n \log(\mathcal{N}_n)\mathrm{VC}(\Pi)}{n_e}}$. By construction of Algorithm 2, $J = \mathcal{O}(1)$. By Lemma D.5, $\chi(A^2) \leq 2\mathcal{N}_n^2$.

Rademacher complexity bounds for (iii) Since (iii) does not depend on estimators, the bound for (iii) follows from the same argument in Theorem D.1. Recall the definitions of $\chi(A^2), \mathcal{C}_n^2(g)$ I used in Theorem D.1. Following the proof of Theorem D.1 (Paragraph "Symmetrization and proper cover"), I can write

$$(iii) \leq \sum_{g\in\{1,\cdots,\chi(A^2)\}} \mathbb{E}\Big[\mathbb{E}_{\sigma,\tilde{\varepsilon}}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^2(g)} R_i\tilde{\varepsilon}_i\frac{I_i(\pi)}{e_i(\pi)}\Big|\Big]|A, Z\Big].$$

I can now bound $\mathbb{E}_{\sigma,\tilde{\varepsilon}}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^2(g)} R_i\tilde{\varepsilon}_i\frac{I_i(\pi)}{e_i(\pi)}\Big|\Big]$ directly with Lemma D.8, with $\tilde{\varepsilon}_i$ in lieu of Ω_i in Lemma D.8 and $I_i(\pi)/e_i(\pi)$ in lieu of $g_i(\cdot)$ in Lemma D.8, with upper bound $U_n = 2/(\gamma\delta_n)$. Following the same argument as in Equation (D.8)

$$\sum_{g\in\{1,\cdots,\chi(A^2)\}} \mathbb{E}\Big[\mathbb{E}_{\sigma,\tilde{\varepsilon}}\Big[\sup_{\pi\in\Pi}\Big|\frac{1}{n_e}\sum_{i\in\mathcal{C}_n^2(g)} R_i\tilde{\varepsilon}_i\frac{I_i(\pi)}{e_i(\pi)}\Big|\Big]|A, Z\Big] \leq c'\frac{\Gamma\sqrt{\chi(A_n^2)}}{\gamma\delta_n}\sqrt{\frac{\mathcal{N}_n\log(\mathcal{N}_n)\mathrm{VC}(\Pi)}{n_e}}.$$

By Lemma D.5, $\chi(A^2) \leq 2\mathcal{N}_n^2$, for a universal constant $c' < \infty$.

Rademacher complexity bounds for (iv) The bound for (iv) follows verbatim as the bound for (ii), where, here, instead of conditioning on $\hat{e}_{i\in F_k^j}$ as in Equation (D.19), I condition on $\hat{m}_{i\in F_k^j}$. This is omitted for space constraints. The proof completes.

D.2.4 Proof of Theorem 4.1

Define $W_{A,Z}^{tr}(\pi) = \frac{1}{n}\sum_{i=1}^n m\Big(\pi(X_i), T_i(\pi), Z_i, |N_i|\Big)1\Big\{|N_i| \leq \log_\gamma(\kappa_n)\Big\}$ the trimmed version of welfare. Following Corollary 1,

$$\begin{aligned}\mathbb{E}\Big[\sup_{\pi\in\Pi_n} W_{A,Z}(\pi) - W_{A,Z}(\hat{\pi}_{\kappa_n}^{tr})\Big|A, Z\Big] &\leq 2\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|W_{A,Z}(\pi) - W_n^{tr}(\pi)\Big||A, Z\Big] \\ &\leq 2\mathbb{E}\Big[\sup_{\pi\in\Pi}\Big|W_{A,Z}^{tr}(\pi) - W_n^{tr}(\pi)\Big||A, Z\Big] + 2\sup_{\pi\in\Pi}\Big|W_{A,Z}^{tr}(\pi) - W_{A,Z}(\pi)\Big|.\end{aligned} \tag{D.22}$$

The bounds for the first component in the right-hand side of Equation (D.22) follows verbatim the proof of Theorem D.1, since $\mathbb{E}[W_n^{tr}(\pi)|A, Z] = W_{A,Z}^{tr}(\pi)$, with the difference that the overlap constant is $\gamma^{\log_\gamma(\kappa_n)+1}$ under Assumption 2.2 (i). For the second component,

$$\Big|W_{A,Z}^{tr}(\pi) - W_{A,Z}(\pi)\Big| \leq \frac{1}{n}\sum_{i=1}^n m\Big(\pi(X_i), T_i(\pi), Z_i, |N_i|\Big)\Big(1 - 1\Big\{|N_i| \leq \log_\gamma(\kappa_n)\Big\}\Big). \tag{D.23}$$

Note that (D.23) $= \mathcal{O}\Big(\frac{1}{n}\sum_{i=1}^n 1\Big\{|N_i| > \log_\gamma(\kappa_n)\Big\}\Big)$, by Assumption 2.3 (D) (since the third moment is bounded, also the first moment is uniformly bounded) and Holder's inequality.

D.2.5 Proof of Proposition 4.4

By Lemma 4.3, $\mathbb{E}[\tilde{W}_n(\pi, m^c, e)|A, Z, A', Z'] = W_{A',Z'}(\pi)$. Therefore, the same argument of the proof of Theorem D.1 holds, with the difference that the Lipschitz constant in the proof of Theorem D.1 now multiplies by $\bar{L}_{A,Z,n}$.

D.3 Lemmas

Lemma D.2. *The following holds: $\chi(A_n) \leq \chi(A_n^M) \leq M\mathcal{N}_n^M$ for all $n \geq 1$.*

Proof of Lemma D.2. The first inequality follows by Definition D.3. The second inequality follows by Brook's Theorem (Brooks, 1941), since the maximum degree under A_n^M is bounded by $\mathcal{N}_n + \mathcal{N}_n \times \mathcal{N}_n + \cdots + \prod_{s=1}^M \mathcal{N}_n \leq M\mathcal{N}_n^M$. $\square$

Lemma D.3. *For $i \in \{1, \cdots, n\}$ consider functions $f_i : \mathcal{T}_n \mapsto [-U_n, U_n]$ for some $U_n > 0$, and $\mathcal{T}_n \subseteq \mathbb{Z}$. Then for any $i \in \{1, \cdots, n\}, n \geq 1$, $f_i(t)$ is $2U_n$-Lipschitz in t.*

Proof of Lemma D.3. For any $t, t' \in \mathbb{Z}$, $\Big|f_i(t) - f_i(t')\Big| \leq 2U_n$ for $t \neq t'$, by the triangular inequality. Since $\mathcal{T}_n \subseteq \mathbb{Z}$ is discrete, $\Big|f_i(t) - f_i(t')\Big| \leq 2U_n|t - t'|$. $\square$

Lemma D.4. *For any $i \in \{1, \cdots, n\}$, let $X_i \in \mathcal{X}$ be an arbitrary random variable and $\mathcal{F}$ a class of uniformly bounded functions with envelope $\bar{F}$. Let $\Omega_i|X_1, \cdots, X_n$ be random variables independently but not necessarily identically distributed, where $\Omega_i \geq 0$ is a scalar. Assume that for some $u > 0$, $\mathbb{E}[\Omega_i^{2+u}|Z] < B, \quad \forall i \in \{1, \cdots, n\}$. In addition, assume that for any fixed points $x_1^n \in \mathcal{X}^n$, for some $V_n \geq 0$, for all $n \geq 1$, $\int_0^{2\bar{F}} \sqrt{\log\Big(\mathcal{M}_1\Big(\eta, \mathcal{F}(x_1^n)\Big)\Big)} d\eta < \sqrt{V_n}$. Let σ_i be i.i.d Rademacher random variables independent of $(\Omega_i)_{i=1}^n, (X_i)_{i=1}^n$. Then for a constant $0 < C_{\bar{F}} < \infty$ that only depend on $\bar{F}$ and u, for all $n \geq 1$*

$$\int_0^\infty \mathbb{E}\Big[\sup_{f \in \mathcal{F}} \Big|\frac{1}{n}\sum_{i=1}^n \sigma_i f(X_i) 1\{\Omega_i > \omega\}\Big| |X_1, \cdots, X_n\Big] d\omega \leq C_{\bar{F}} \sqrt{\frac{BV_n}{n}}.$$

Proof of Lemma D.4. The proof follows verbatim the proof of Lemma A.5 in Kitagawa and Tetenov (2019), with two small differences that do not affect the argument of the proof: I must control the Rademacher complexity using the Dudley's entropy integral bound (instead of the VC dimension), and Ω_i are independent but not necessarily identically distributed random variables. Given that the argument follows verbatim the one of Lemma A.5 of Kitagawa and Tetenov (2019), the proof is omitted for space constraints.[15] $\square$

[15] The reader may refer to a technical note that collects lemmas from past literature available at dviviano.github.io/projects/note_preliminary_lemmas.pdf for details.

Lemma D.5. *Take any $k \geq 2$. Let $\mathcal{F}_1, \cdots, \mathcal{F}_k$ be classes of bounded functions with VC dimension v and envelope $\bar{F} < \infty$. Let*

$$\mathcal{J}_n = \Big\{ f_1(f_2 + ... + f_k), \quad f_j \in \mathcal{F}_j, \quad j = 1, \cdots, k \Big\}, \quad \mathcal{J}_n(x_1^n) = \Big\{ h(x_1), \cdots, h(x_n); h \in \mathcal{J}_n \Big\}.$$

For arbitrary fixed points $x_1^n \in \mathcal{X}^n$, for any $n \geq 1, k \geq 2, v \geq 1$, $\int_0^{2\bar{F}} \sqrt{\log\Big(\mathcal{M}_1\Big(\eta, \mathcal{J}(x_1^n)\Big)\Big)} d\eta < c_{\bar{F}}\sqrt{k \log(k) v}$ for a constant $c_{\bar{F}} < \infty$ that only depends on $\bar{F}$.

Proof of Lemma D.5. Without loss of generality let $\bar{F} \geq 1$ (since if less than one the envelope is also uniformly bounded by one). Let $\mathcal{F}_{-1,n}(x_1^n) = \{f_2(x_1^n) + ... + f_k(x_1^n), f_j \in \mathcal{F}_j, j = 2, ..., k_n\}$. By Devroye et al. (2013), Theorem 29.6, $\mathcal{M}_1\Big(\eta, \mathcal{F}_{-1,n}(x_1^n)\Big) \leq \prod_{j=2}^k \mathcal{M}_1\Big(\eta/(k-1), \mathcal{F}_j(x_1^n)\Big)$. By Theorem 29.7 in Devroye et al. (2013),

$$\mathcal{M}_1\Big(\eta, \mathcal{J}_n(x_1^n)\Big) \leq \prod_{j=2}^k \mathcal{M}_1\Big(\frac{\eta}{2(k-1)\bar{F}}, \mathcal{F}_j(x_1^n)\Big)\mathcal{M}_1\Big(\frac{\eta}{2\bar{F}}, \mathcal{F}_1(x_1^n)\Big). \tag{D.24}$$

By standard properties of covering numbers, for a generic set $\mathcal{H}$, $\mathcal{N}_1(\eta, \mathcal{H}) \leq \mathcal{N}_2(\eta, \mathcal{H})$. It follows (D.24) $\leq \prod_{j=2}^k \mathcal{M}_2\Big(\frac{\eta}{2(k-1)\bar{F}}, \mathcal{F}_j(x_1^n)\Big)\mathcal{M}_2\Big(\frac{\eta}{2\bar{F}}, \mathcal{F}_1(x_1^n)\Big)$. I now apply a uniform entropy bound for the covering number. By Theorem 2.6.7 of Van Der Vaart and Wellner (1996), we have that for a universal constant $C < \infty$ (that without loss of generality we can assume $C \geq 1$), $\mathcal{M}_2\Big(\frac{\eta}{2(k-1)\bar{F}}, \mathcal{F}_j(x_1^n)\Big) \leq C(v+1)(16e)^{(v+1)}\Big(\frac{2\bar{F}^2(k-1)}{\eta}\Big)^{2v}$ which implies that

$$\begin{aligned}\log\Big(\mathcal{M}_1\Big(\eta, \mathcal{J}_n(x_1^n)\Big)\Big) &\leq \sum_{j=1}^{k_n - 1} \log\Big(\mathcal{M}_2\Big(\frac{\eta}{2\bar{F}(k-1)}, \mathcal{F}_j(x_1^n)\Big)\Big) + \log\Big(\mathcal{M}_2\Big(\frac{\eta}{2\bar{F}}, \mathcal{F}_1(x_1^n)\Big)\Big) \\ &\leq k \log\Big(C(v+1)(16e)^{v+1}\Big) + k2v\log(2C\bar{F}^2(k-1)/\eta).\end{aligned}$$

Since $\int_0^{2\bar{F}} \sqrt{k \log\Big(C(v+1)(16e)^{v+1}\Big) + k_n 2v \log(2C\bar{F}^2(k-1)/\eta)} d\eta \leq c_{\bar{F}}\sqrt{k\log(k)v}$ for a constant $c_{\bar{F}} < \infty$, the proof completes. $\square$

We discuss the Ledoux and Talagrand (2011)'s inequality for the case of interest here.

Lemma D.6. *For all $i \in \{1, \cdots, n\}$, let $\phi_i : \mathbb{R} \mapsto \mathbb{R}$ be such that $|\phi_i(a) - \phi_i(b)| \leq L|a - b|$ for all $a, b \in \mathbb{R}$, with $\phi_i(0) = 0$, and arbitrary $L > 0$. Then, for any $n \geq 1, L > 0$, any $\mathcal{U}_n \subseteq \mathbb{R}^n, \mathcal{K}_n \subseteq \{0,1\}^n$, with $u = (u_1, \cdots, u_n) \in \mathcal{U}_n$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathcal{K}_n$,*

$$\frac{1}{2}\mathbb{E}_\sigma\Big[\sup_{u \in \mathcal{U}_n, \alpha \in \mathcal{K}_n} \Big|\frac{1}{n}\sum_{i=1}^n \sigma_i \phi_i(u_i)\alpha_i\Big|\Big] \leq L\mathbb{E}_\sigma\Big[\sup_{u \in \mathcal{U}_n, \alpha \in \mathcal{K}_n} \Big|\frac{1}{n}\sum_{i=1}^n \alpha_i \sigma_i u_i\Big|\Big].$$

Proof of Lemma D.6. The proof follows closely the one of Theorem 4.12 in Ledoux and Talagrand (2011) while dealing with the additional α vector. We provide here the main argument and refer to Ledoux and Talagrand (2011) for additional details. First, note that if $\mathcal{U}_n$ is unbounded, there will be settings such that the right hand side is infinity and the result trivially holds. Therefore, let $\mathcal{U}_n$ be bounded. We aim to show that

$$\mathbb{E}\Big[\sup_{u\in\mathcal{U}_2,\alpha\in\mathcal{K}_2}\alpha_1 u_1+\sigma_2\phi(u_2)\alpha_2\Big]\leq\mathbb{E}\Big[\sup_{u\in\mathcal{U}_2,\alpha\in\mathcal{K}_2}\alpha_1 u_1+L\sigma_2 u_2\alpha_2\Big]. \tag{D.25}$$

If Equation (D.25), it follows that

$$\mathbb{E}\Big[\sup_{u\in\mathcal{U}_2,\alpha\in\mathcal{K}_2}\alpha_1\phi_1(u_1)\sigma_1+\sigma_2\phi(u_2)\alpha_2|\sigma_1\Big]\leq\mathbb{E}\Big[\sup_{u\in\mathcal{U}_2,\alpha\in\mathcal{K}_2}\alpha_1\phi_1(u_1)\sigma_1+L\sigma_2 u_2\alpha_2\Big|\sigma_1\Big].$$

Because $\sigma_1\phi(u_1)$ simply transforms $\mathcal{U}_2$, and we can iteretively apply this result.

I first prove Equation (D.25). Define for $a,b\in\{0,1\}^2$, $I(u,s,a,b):=\frac{1}{2}\Big(u_1a_1+a_2\phi(u_2)\Big)+\frac{1}{2}\Big(s_1b_1-b_2\phi(s_2)\Big)$. I want to show that the right hand side in Equation (D.25) is larger than $I(u,s,a,b)$ for all $u,s\in\mathcal{U}_2$ and $a,b\in\{0,1\}^2$. Since I am taking the supremum of $I(u,s,a,b)$ over u,s,a,b, I can assume without loss of generality (as in Ledoux and Talagrand, 2011)

$$u_1a_1+a_2\phi(u_2)\geq s_1b_1+b_2\phi(s_2),\quad s_1b_1-b_2\phi(s_2)\geq u_1a_1-a_2\phi(u_2). \tag{D.26}$$

I can now define four quantities of interest

$$q_1=b_1s_1-b_2\phi(s_2),\quad q_2=b_1s_1-Ls_2b_2,\quad q_1'=a_1u_1+La_2u_2,\quad q_2'=a_1u_1+a_2\phi(u_2).$$

I consider four different cases, similarly to Ledoux and Talagrand (2011) and argue that for any value of $(a_1,a_2,b_1,b_2)\in\{0,1\}^4$, $2I(u,s,a,b)=q_1+q_2'\leq q_1'+q_2$.

Case 1 Start from the case $a_2u_2,s_2b_2\geq 0$. We know that $\phi(0)=0$, so that $|b_2\phi(s_2)|\leq Lb_2s_2$. Now assume that $a_2u_2\geq b_2s_2$. In this case $q_1-q_2=Lb_2s_2-b_2\phi(s_2)\leq La_2u_2-a_2\phi(u_2)=q_1'-q_2'$ since $|a_2\phi(u_2)-b_2\phi(s_2)|\leq L|a_2u_2-b_2s_2|=L(a_2u_2-b_2s_2)$. To see why this last claim holds, note that for $a_2,b_2=1$, then the results hold by the condition $a_2u_2\geq b_2s_2$ and Lipschitz continuity. If instead $a_2=1,b_2=0$, the claim trivially holds. While the case $a_2=0,b_2=1$, then it must be that $s_2=0$ since we assumed that $a_2u_2\geq 0,b_2s_2\geq 0$ and $a_2u_2\geq b_2s_2$. Thus $q_1-q_2\leq q_1'-q_2'$. If instead $b_2s_2\geq a_2u_2$, then use $-\phi$ instead of ϕ and switch the roles of s,u giving a similar proof.

Case 2 Let $a_2u_2\leq 0,b_2s_2\leq 0$. Then the proof is the same as Case 1, switching the signs where necessary.

Case 3 Let $a_2u_2\geq 0,b_2s_2\leq 0$. Then $a_2\phi(u_2)\leq La_2u_2$, since $a_2\in\{0,1\}$ and by Lipschitz properties of ϕ, $-b_2\phi(s_2)\leq -b_2Ls_2$ so that $a_2\phi(u_2)-b_2\phi(s_2)\leq a_2Lu_2-b_2Ls_2$.

Case 4 Let $a_2u_2\leq 0,b_2s_2\geq 0$. Then the claim follows symmetrically to Case 3.

The conclusion of the proof follows verbatim the one in Ledoux and Talagrand (2011). □

Below, I use the same notation as in the main text, while I also introduce an additional random variable Ω_i with $\Omega=(\Omega_1,\cdots,\Omega_n)$.

Lemma D.7. *Let* Π, Π' *be two function classes, each with VC dimension* v, *and* $\pi : \mathcal{X} \mapsto \{0,1\}$ *for any* $\pi \in \Pi, \Pi'$. *For* $i \in \{1, \cdots, n\}$, *take arbitrary* $(X_{k\in N_i}, X_i), X_i \in \mathcal{X}, \Omega_i \in \mathbb{R}, R_i \in \{0,1\}$, *adjacency matrix* A, *and functions* $f_i : \mathbb{Z} \mapsto [-U_n, U_n]$, *for a positive constant* $U_n > 0$. *Assume that* $\mathbb{E}[|\Omega_i|^3 | (R_i)_{i=1}^n, (X_i)_{i=1}^n, A] < B$, *for some* $B < \infty$, *and* $(\Omega_i)_{i=1}^n | (R_i)_{i=1}^n, (X_i)_{i=1}^n, A$ *are independent but not necessarily identically distributed. Let* $\sigma_1, \cdots, \sigma_n$ *be i.i.d. Rademacher random variables, independent of* $\Big[\Big(X_i, R_i, \Omega_i\Big)_{i=1}^n, A\Big]$. *Then for a universal constant* $c_0 < \infty$, *for any* $n \geq 1$, $v = \mathrm{VC}(\Pi) = \mathrm{VC}(\Pi')$

$$\mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big)\pi_1(X_i)\sigma_i\Omega_i\Big|\Big] \leq c_0 U_n \sqrt{vB\mathcal{N}_n \log(\mathcal{N}_n)\sum_{i=1}^n R_i}. \tag{D.27}$$

Proof of Lemma D.7. First, note that since $R_i \in \{0,1\}$, and we take the expectation conditional on $(R_i)_{i=1}^n$, we can interpret the sum in Equation (D.27) as a sum over elements $\sum_{i=1}^n R_i$ many elements. Also, note that from Lemma D.3, we have that $f_i(t)$ is $2U_n$-Lipschitz in t.

First decomposition First, we add and subtract the value of the function $f_i(0)$ at zero. The left hand side in Equation (D.27) equals

$$\begin{aligned}
&\mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\sigma_i\Big(f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big) - f_i(0) + f_i(0)\Big)\Omega_i\pi_1(X_i)\Big|\Big] \\
&\leq \underbrace{\mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\sigma_i\Big(f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big) - f(0)\Big)\Omega_i\pi_1(X_i)\Big|\Big]}_{(1)} + \underbrace{\mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi_1\in\Pi}\Big|\sum_{i=1}^n R_i\sigma_i f_i(0)\Omega_i\pi_1(X_i)\Big|\Big]}_{(2)}.
\end{aligned} \tag{D.28}$$

First, I bound (1). I write

$$\begin{aligned}
(1) &= \mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\sigma_i\Big(f_i\Big(\sum_{kN_i}\pi_2(X_k)\Big) - f_i(0)\Big)|\Omega_i|\mathrm{sign}(\Omega_i)\pi_1(X_i)\Big|\Big] \\
&= \mathbb{E}_{\Omega,\tilde\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\tilde\sigma_i\Big(f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big) - f_i(0)\Big)|\Omega_i|\pi_1(X_i)\Big|\Big]
\end{aligned} \tag{D.29}$$

where $\tilde\sigma_i = \mathrm{sign}(\Omega_i)\sigma_i$ which are *i.i.d.* Rademacher random variables independent of $(\Omega_i, X_i, R_i)_{i=1}^n, A$, since $P(\tilde\sigma_i = 1|\Omega) = P(\sigma_i\mathrm{sign}(\Omega_i) = 1|\Omega) = 1/2$. Using the fact that $|\Omega_i| \geq 0$, I have

$$\begin{aligned}
\text{(D.29)} &= \mathbb{E}_{\Omega,\tilde\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\tilde\sigma_i\Big(f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big) - f(0)\Big)\int_0^\infty 1\{|\Omega_i| > \omega\}d\omega\pi_1(X_i)\Big|\Big] \\
&\leq \mathbb{E}_{\Omega,\tilde\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\int_0^\infty\Big|\sum_{i=1}^n R_i\tilde\sigma_i\Big(f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big) - f_i(0)\Big)1\{|\Omega_i| > \omega\}\pi_1(X_i)\Big|d\omega\Big] \\
&\leq \int_0^\infty \mathbb{E}_{\Omega,\tilde\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\tilde\sigma_i\Big(f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big) - f_i(0)\Big)1\{|\Omega_i| > \omega\}\pi_1(X_i)\Big|\Big]d\omega.
\end{aligned} \tag{D.30}$$

Next, I use the law of iterated expectation to first take the expectation over $\tilde{\sigma}$ (conditional on Ω) and then take the expectation over Ω. I also divide and multiplied by U_n. I obtain

$$(\text{D.30}) \leq U_n \int_0^\infty \mathbb{E}_\Omega\Big[\mathbb{E}_{\tilde{\sigma}}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i \frac{1}{U_n}\Big(f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big) - f_i(0)\Big)1\{|\Omega_i| > \omega\}\pi_1(X_i)\Big|\Big]\Big]d\omega. \tag{D.31}$$

Lipschitz property Let $\phi_i(t) = \frac{1}{U_n}(f_i(t) - f_i(0))$. Here, ϕ_i is Lipschitz in t, with Lipschitz constant equal to 1. In addition, $\phi_i(0) = 0$. By Lemma D.6[16],

$$\begin{aligned}
&\mathbb{E}_{\tilde{\sigma}}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i \frac{1}{U_n}\Big(f_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big) - f_i(0)\Big)1\{|\Omega_i| > \omega\}\pi_1(X_i)\Big|\Big] \\
&\leq 2\mathbb{E}_{\tilde{\sigma}}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big)1\{|\Omega_i| > \omega\}\pi_1(X_i)\Big|\Big].
\end{aligned} \tag{D.32}$$

I can therefore write

$$(\text{D.31}) \leq 2U_n \int_0^\infty \mathbb{E}_{\Omega,\tilde{\sigma}}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big)1\{|\Omega_i| > \omega\}\pi_1(X_i)\Big|\Big]d\omega.$$

Function reparametrization I now consider a reparametrization of the function class. Define $\tilde{X}_i \in \mathcal{X}^{\mathcal{N}_n} = (X_i, X_{k\in N_i}, \emptyset, \cdots, \emptyset)$, where for the entries $h > |N_i| + 1$, $\tilde{X}_i^{(h)} = \emptyset$, denoting the h^{th} entry of $\tilde{X}_i$. Without loss of generality, let $\pi(\emptyset) = 0$. Define $\pi_j \in \Pi_j$ a function class of the form $\pi_j(\tilde{X}_i) = \pi(\tilde{X}_i^{(j)}), \pi \in \Pi'$ for $j > 1$ and $\pi_1(\tilde{X}_i) = \pi(\tilde{X}_i^{(1)}), \pi \in \Pi$, i.e., equal to π applied to the j^{th} entry of the vector $\tilde{X}_i$. Since this is a trivial reparametrization, $\mathrm{VC}(\Pi_j) = \mathrm{VC}(\Pi)$ ($= \mathrm{VC}(\Pi')$ by assumption) for all $j \in \{1, \cdots, \mathcal{N}_n\}$.[17] I can write

$$\begin{aligned}
&U_n \int_0^\infty \mathbb{E}_{\Omega,\tilde{\sigma}}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi'}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i\Big(\sum_{k\in N_i}\pi_2(X_k)\Big)1\{|\Omega_i| > \omega\}\pi_1(X_i)\Big|\Big]d\omega \\
&\leq U_n \int_0^\infty \mathbb{E}_{\Omega,\tilde{\sigma}}\Big[\sup_{\tilde{\pi}_1\in\Pi_1,\cdots,\tilde{\pi}_{\mathcal{N}_n}\in\Pi_{\mathcal{N}_n}}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i\Big(\sum_{k=1}^{\mathcal{N}_n-1}\tilde{\pi}_{k+1}(\tilde{X}_i)\Big)1\{|\Omega_i| > \omega\}\tilde{\pi}_1(\tilde{X}_i)\Big|\Big]d\omega \\
&= U_n \int_0^\infty \mathbb{E}_{\Omega,\tilde{\sigma}}\Big[\sup_{\tilde{\pi}\in\tilde{\Pi}_n}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i\tilde{\pi}(\tilde{X}_i)1\{|\Omega_i| > \omega\}\Big|\Big]d\omega
\end{aligned}$$

where $\tilde{\Pi}_n = \Big\{\pi_1\Big(\sum_{j=2}^{\mathcal{N}_n-1}\pi_{j+1}\Big), \pi_j \in \Pi_j, j = 1, \cdots, \mathcal{N}_n\Big\}$. I now apply Lemma D.5, using the fact that $\mathrm{VC}(\Pi_j) = \mathrm{VC}(\Pi) = \mathrm{VC}(\Pi')$, for any $j \in \{1, \cdots, \mathcal{N}_n\}$. By Lemma D.5,

[16] Conditional on X, A, Ω, I invoke Lemma D.6 with $(\pi_1(X_i)1\{|\Omega_i| > \omega\})_{i=1}^n$ in lieu of $(\alpha_1, \cdots, \alpha_n) \in \mathcal{K}_n \subseteq \{0,1\}^n$ in the statement of Lemma D.6, since $\pi_1(X_i)1\{|\Omega_i| > \omega\}$ is binary. Here $(\sum_{k\in N_i}\pi_2(X_k))_{i=1}^n$ is in lieu of $(u_1, \cdots, u_n) \in \mathcal{U}_n$ in Lemma D.6. The spaces $\mathcal{K}_n, \mathcal{U}_n$ in Lemma D.6, here are those defined (given Ω, X, A), by $\pi_1(X_i)1\{|\Omega_i| > \omega\}, \pi_1 \in \Pi$ and $(\sum_{k\in N_i}\pi_2(X_k))_{i=1}^n, \pi_2 \in \Pi'$, respectively.

[17] See e.g., Theorem 29.4 in Devroye et al. (2013).

for any $n \geq 1$, the Dudley's integral of the function class $\tilde{\Pi}_n$ is uniformly bounded by $C\sqrt{\mathcal{N}_n \log(\mathcal{N}_n)\mathrm{VC}(\Pi)}$, for a finite universal constant C. By Lemma D.4, since I am summing over $\sum_{i=1}^n R_i$ elements (conditional on $(R_1, \cdots, R_n)$), for a universal constant $\bar{C}' < \infty$

$$U_n \int_0^\infty \mathbb{E}_{\Omega,\tilde{\sigma}}\Big[\sup_{\tilde{\pi}\in\tilde{\Pi}_n}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i\tilde{\pi}(\tilde{X}_i)1\{|\Omega_i| > \omega\}\Big|\Big]d\omega \quad \leq \bar{C}' U_n \sqrt{B\mathcal{N}_n \mathrm{VC}(\Pi)\log(\mathcal{N}_n)\sum_{i=1}^n R_i}.$$

Term (2) Next, I bound the term (2) in Equation (D.28). Similar to (1),

$$\begin{aligned}\mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi\in\Pi}\Big|\sum_{i=1}^n R_i\sigma_i f_i(0)\Omega_i\pi(X_i)\Big|\Big] &\leq U_n\mathbb{E}_{\Omega,\tilde{\sigma}}\Big[\sup_{\pi\in\Pi}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i|\frac{f_i(0)}{U_n}\Omega_i|\pi(X_i)\Big|\Big] \\ &\leq U_n\int_0^\infty \mathbb{E}_{\Omega,\tilde{\sigma}}\Big[\sup_{\pi\in\Pi}\Big|\sum_{i=1}^n R_i\tilde{\sigma}_i 1\{|f_i(0)\Omega_i|/U_n > \omega\}\pi(X_i)\Big|\Big]d\omega.\end{aligned}$$

Since Π has finite VC dimension, by Theorem 2.6.7 of Van Der Vaart and Wellner (1996) (the argument is the same as in Lemma D.5), $\int_0^2 \sqrt{\mathcal{M}_1(\eta, \Pi(x_1^n))}d\eta < C\sqrt{\mathrm{VC}(\Pi)}$ for a universal constant C, and for any $x_1^n \in \mathcal{X}^n$. Since $\mathbb{E}_\Omega[|f_i(0)\Omega_i/U_n|^3] \leq B$ ($f_i(0)/U_n \in [-1,1]$) we can apply Lemma D.4, with $|f_i(0)\Omega_i|/U_n$ in lieu of $|\Omega_i|$ in Lemma D.4, and obtain

$$U_n\int_0^\infty \mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi\in\Pi}\Big|\sum_{i=1}^n R_i\sigma_i 1\{|f_i(0)\Omega_i|/U_n > \omega\}\pi(X_i)\Big|\Big]d\omega \leq C' U_n\sqrt{B\mathrm{VC}(\Pi)\sum_{i=1}^n R_i}$$

for a universal constant $C' < \infty$. The proof completes. □

The following lemma is a direct corollary of Lemma D.7.

Lemma D.8. *Let $\pi \in \Pi$, be a function class, with $\pi : \mathcal{X} \mapsto \{0,1\}$. For $i \in \{1, \cdots, n\}$, take arbitrary $(X_{k\in N_i}, X_i), X_i \in \mathcal{X}, \Omega_i \in \mathbb{R}, R_i \in \{0,1\}$, adjacency matrix A, and functions $g_i : \mathbb{Z}\times\{0,1\} \mapsto [-U_n, U_n]$, for a positive constant $U_n > 0$. Assume that $\mathbb{E}[|\Omega_i|^3|(R_i)_{i=1}^n, (X_i)_{i=1}^n, A] < B$, for some $B < \infty$, and $(\Omega_i)_{i=1}^n|(R_i)_{i=1}^n, (X_i)_{i=1}^n, A$ are independent but not necessarily identically distributed. Let $\sigma_1, \cdots, \sigma_n$ be i.i.d. Rademacher random variables, independent of $\Big[\Big(X_i, R_i, \Omega_i\Big)_{i=1}^n, A\Big]$. Then for a universal constant $c_0 < \infty$, for any $n \geq 1$*

$$\mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi\in\Pi}\Big|\sum_{i=1}^n R_i g_i\Big(\sum_{k\in N_i}\pi(X_k), \pi(X_i)\Big)\sigma_i\Omega_i\Big|\Big] \leq c_0 U_n\sqrt{\mathrm{VC}(\Pi)B\mathcal{N}_n\log(\mathcal{N}_n)\sum_{i=1}^n R_i}. \tag{D.33}$$

Proof of Lemma D.8. By Lemma D.3, $g_i(t,1), g_i(t,0)$ are $2U_n$-Lipschitz in t. It follows

$$\begin{aligned}&\mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi\in\Pi}\Big|\sum_{i=1}^n R_i g_i\Big(\sum_{k\in N_i}\pi(X_k), \pi(X_i)\Big)\sigma_i\Omega_i\Big|\Big] \\ &\leq \mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi\in\Pi}\Big|\sum_{i=1}^n R_i g_i\Big(\sum_{k\in N_i}\pi(X_k), 1\Big)\pi(X_i)\sigma_i\Omega_i\Big|\Big] + \mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi\in\Pi}\Big|\sum_{i=1}^n R_i g_i\Big(\sum_{k\in N_i}\pi(X_k), 0\Big)(1-\pi(X_i))\sigma_i\Omega_i\Big|\Big]\end{aligned} \tag{D.34}$$

It follows

$$(\text{D.34}) \leq \mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi_1\in\Pi,\pi_2\in\Pi}\Big|\sum_{i=1}^n R_i g_i\Big(\sum_{k\in N_i}\pi_2(X_k),1\Big)\pi_1(X_i)\sigma_i\Omega_i\Big|\Big]$$
$$+\mathbb{E}_{\Omega,\sigma}\Big[\sup_{\pi_1'\in\Pi,\pi_2'\in\Pi}\Big|\sum_{i=1}^n R_i g_i\Big(\sum_{k\in N_i}\pi_2'(X_k),0\Big)(1-\pi_1'(X_i))\sigma_i\Omega_i\Big|\Big].$$

By Lemma 29.4 in Devroye et al. (2013), the VC dimension of the function class $1-\pi, \pi\in\Pi$ equals the VC(Π). By Lemma D.7 each term in Equation (D.34) is bounded by $CU_n\sqrt{\text{VC}(\Pi)B\mathcal{N}_n\log(\mathcal{N}_n)\sum_{i=1}^n R_i}$, for a universal constant $C<\infty$. □

Lemma D.9. *Let K^* be as in Algorithm 2 (Equation 24). Then $K^*\leq\chi(A^2)$ almost surely.*

Proof of Lemma D.9. To prove the claim it suffices to show that a partition such that the constraints in Equation (24) holds exists, and such a partition has size at most $\chi(A^2)$, for all possible realizations of $R=(R_1,\cdots,R_n)$. As a first step, observe that for fixed K, binary variables $G_{j,k}\in\{0,1\}, j\in\{1,\cdots,n\}, k\in\{1,\cdots,K\}$, with $\sum_{k=1}^K G_{j,k}=1\forall j\in\{1,\cdots,n\}$,

$$\sum_{k=1}^K\sum_{j=1}^n 1\{j\not\in N_i \text{ and } N_i\cap N_j=\emptyset\}G_{j,k}G_{i,k}=0 \text{ implies } \sum_{k=1}^K\sum_{j=1}^n R_iR_j1\{j\in\mathcal{I}_i\}G_{j,k}G_{i,k}=0.$$

Namely, $\sum_{k=1}^K\sum_{j=1}^n 1\{j\not\in N_i \text{ and } N_i\cap N_j=\emptyset\}G_{j,k}G_{i,k}=0$ is a stricter constraint than $\sum_{k=1}^K\sum_{j=1}^n R_iR_j1\{j\in\mathcal{I}_i\}G_{j,k}G_{i,k}=0$, in Equation (24), for all $R_1,\cdots,R_n, R_i\in\{0,1\}$. I can therefore bound the solution to the optimization problem in Equation (24) as follows

$$\begin{aligned}K^*\leq \arg\min_{K\in\mathbb{Z}}\min_{G\in\{0,1\}^{n\times K}} & K\\ \text{such that } & \sum_{k=1}^K\sum_{j=1}^n 1\{j\not\in N_i, N_i\cap N_j=\emptyset\}G_{j,k}G_{i,k}=0, \text{ and } \sum_{k=1}^K G_{i,k}=1\forall i.\end{aligned}\tag{D.35}$$

The right-hand side in Equation (D.35) equals $\chi(A^2)$ by definition of smallest proper cover. □

D.3.1 Identification

Proof of Lemma 2.1. Let $e\Big(\pi(X_i),T_i(\pi),Z_{k\in N_i},R_{k\in N_i},Z_i,|N_i|\Big)=e_i(\pi), I_i(\pi)=1\{T_i(\pi)=T_i,\pi(X_i)=D_i\}$. Under Assumption 2.1, I can write

$$\mathbb{E}\Big[R_i\frac{I_i(\pi)}{e_i(\pi)}Y_i\Big|A,Z\Big]=\mathbb{E}\Big[R_i\frac{I_i(\pi)}{e_i(\pi)}r\Big(\pi(X_i),T_i(\pi),Z_i,|N_i|,\varepsilon_i\Big)\Big|A,Z\Big].\tag{D.36}$$

Under Assumption 2.2 (i) and Assumption 2.3 (A),

$$(\text{D.36})=\mathbb{E}\Big[\frac{R_iI_i(\pi)}{e_i(\pi)}|A,Z\Big]\times\mathbb{E}\Big[r\Big(\pi(X_i),T_i(\pi),Z_i,|N_i|,\varepsilon_i\Big)\Big|A,Z\Big].$$

By Assumption 2.2 (i), $\mathbb{E}\Big[\frac{R_iI_i(\pi)}{e_i(\pi)}|A,Z\Big]=\mathbb{E}\Big[R_i\mathbb{E}\Big[\frac{I_i(\pi)}{e_i(\pi)}|A,Z,(R_i)_{j\neq i},R_i=1\Big]\Big]=\frac{n_e}{n}$. □

Lemma D.10. *Let Assumptions 2.1, 2.2 hold. Then*

$$\frac{1}{n_e}\sum_{i=1}^n \mathbb{E}\Big[R_i\frac{1\{T_i(\pi)=T_i, d=D_i\}}{e^c\Big(\pi(X_i),T_i(\pi),Z_{k\in N_i},R_{k\in N_i},Z_i,|N_i|\Big)}\Big(Y_i - m^c\Big(\pi(X_i),T_i(\pi),Z_i,|N_i|\Big)\Big)\Big|A,Z\Big]$$
$$+\frac{1}{n_e}\sum_{i=1}^n \mathbb{E}\Big[R_i m^c\Big(\pi(X_i),T_i(\pi),Z_i,|N_i|\Big)\Big|A,Z\Big] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\Big[r\Big((\pi(X_i),T_i(\pi),Z_i,|N_i|,\varepsilon_i\Big)\Big|A,Z\Big]$$

if either $e^c=e$ or (and) Assumption 2.3 (B) holds with $m^c=m$.

Proof of Lemma D.10. Define $e_i^c(\pi) = e^c\Big(\pi(X_i),T_i(\pi),Z_{k\in N_i},R_{k\in N_i},Z_i,|N_i|\Big)$, $I_i(\pi) = 1\{T_i(\pi) = T_i, \pi(X_i) = D_i\}$, $m_i^c = m^c(\pi(X_i),T_i(\pi),Z_i,|N_i|)$. Whenever $e^c=e$, the result directly follows from Lemma 2.1. Let now $m^c=m$ and Assumption 2.3 (B) hold. Then (since the indicators R are independent of ε by Assumption 2.3)

$$\mathbb{E}\Big[\frac{R_iI_i(\pi)}{e_i^c(\pi)}\Big(Y_i-m_i^c(\pi)\Big)\Big|A,Z\Big] = \mathbb{E}\Big[R_i\frac{I_i(\pi)}{e_i^c(\pi)}\Big(r\Big(\pi(X_i),T_i(\pi),Z_i,|N_i|,\varepsilon_i\Big)-m_i(\pi)\Big)\Big|A,Z\Big]$$
$$=\mathbb{E}\Big[R_i\frac{I_i(\pi)}{e_i^c(\pi)}\Big|A,Z\Big]\times\mathbb{E}\Big[\Big(r\Big(\pi(X_i),T_i(\pi),Z_i,|N_i|,\varepsilon_i\Big)-m_i(\pi)\Big)\Big|A,Z\Big]=0.$$

By Assumption 2.2 (iii), $\frac{1}{n_e}\sum_{i=1}^n\mathbb{E}\Big[R_im_i(\pi)\Big|A,Z\Big] = \frac{1}{n}\sum_{i=1}^n m(\pi(X_i),T_i(\pi),Z_i,|N_i|)$. □

Proof of Lemma 4.3. Let $L_i = L(Z_i,Z_{k\in N_i},|N_i|)$ and similarly $L_i' = L'(Z_i,Z_{k\in N_i},|N_i|)$. Let $T_i', Z_i', |N_i|'$ be the neighbors' exposure, covariates and number of neighbors of i in the target population. Following Lemma D.10, by exogeneity of $(R_1,\cdots,R_n)$ (Assumption 2.3 (A))

$$R_i\mathbb{E}\Big[\frac{I_i(\pi)}{e_i(\pi)}\Big(Y_i-m_i^c(\pi)\Big)+m_i^c(\pi)\Big|A,Z,R_1,\cdots,R_n\Big] = R_i\mathbb{E}\Big[r\Big((\pi(X_i),T_i(\pi),Z_i,|N_i|,\varepsilon_i\Big)\Big|A,Z\Big]$$
$$= R_im\Big(\pi(X_i),T_i(\pi),Z_i,|N_i|\Big).$$

where the last equality follows from Assumption 2.3 (B). Therefore, it follows that

$$\mathbb{E}\Big[\tilde{W}_n(\pi,m^c,e)\Big|A,Z\Big] = \frac{1}{n}\sum_{i=1}^n\frac{L_i'}{L_i}m\Big(\pi(X_i),T_i(\pi),Z_i,|N_i|\Big) = \frac{1}{n}\sum_{i=1}^n m\Big(\pi(X_i'),T_i'(\pi).Z_i',|N_i|'\Big),$$

The last equality follows by construction of L_i', L_i. $\mathcal{S}_n(A',Z')\subseteq\mathcal{S}_n(A,Z)$ guarantees that there are no individuals in the target population outside the sample population's support. □

D.4 Proofs for "Additional extensions"

D.4.1 Proof of Proposition B.3

Denote $\mathbb{E}_\pi[\cdot]$ the expectation conditional on $\Big\{D_i=\pi(X_i)\Big\}_{i=1}^n$, let $R=(R_i)_{i=1}^n$. We have

$$\mathbb{E}_\pi\Big[r\Big(S_i,\sum_{k\in N_i}S_k,Z_i,|N_i|,\varepsilon_i\Big)\Big|A,Z\Big] = \mathbb{E}\Big[r\Big(S_i(\pi),\sum_{k\in N_i}S_k(\pi),Z_i,|N_i|,\varepsilon_i\Big)\Big|A,Z,R\Big], \tag{D.37}$$

where $S_i(\pi) = h_\theta\Big(\pi(X_i), \sum_{k \in N_i} \pi(X_k), Z_i, |N_i|, \nu_i\Big)$. It follows that Equation (D.37) equals

$$\sum_{s \in \{0,\cdots,|N_i|\}} \underbrace{\mathbb{E}\Big[r(d,s,Z_i,|N_i|,\varepsilon_i)\Big|S_i(\pi) = d, \sum_{k \in N_i} S_k(\pi) = s, Z, A\Big]}_{(i)} \times \underbrace{P\Big(S_i(\pi) = d, \sum_{k \in N_i} S_k(\pi) = s\Big|A, Z, R\Big)}_{(ii)}.$$

Since $(\varepsilon_j)_{j=1}^n \perp \Big(Z, A, (\varepsilon_{D_j}, \nu_j, R_j)_{j=1}^n\Big)$, I can show $(i) = \mathbb{E}\Big[r(d,s,Z_i,|N_i|,\varepsilon_i)\Big|S_i = d, \sum_{k \in N_i} S_k = s, Z, A, R\Big]$. Consider now (ii). Observe that by indepedence and exogeneity of $(\nu_j)_{j=1}^n$,

$$(ii) = P\Big(S_i(\pi) = d\Big|A, Z, R\Big) \times \sum_{u_1,\cdots,u_l : \sum_v u_v = s} \prod_{k=1}^{|N_i|} P\Big(S_{N_i^{(k)}}(\pi) = u_k\Big|A, Z, R\Big).$$

Using exogeneity of ν_i, I have

$$P\Big(S_i(\pi) = d\Big|A, Z, R\Big) = P\Big(S_i = d\Big|Z_i, |N_i|, D_i = \pi(X_i), \sum_{k \in N_i} D_k = \sum_{k \in N_i} \pi(X_k), Z_{k \in N_i}, Z_i\Big).$$

Similar reasoning also applies to neighbors' selected treatments, omitted for brevity.

D.4.2 Proof of Proposition B.1

To show that Proposition B.1 I need to show that (i) the VC dimension of $\tilde{\Pi}_n$ is at most VC(Π) up-to a constant factor; (ii) overlap holds for any class of policy $\pi \in \tilde{\Pi}_n$, namely $e_i(\pi) \in (\gamma\delta_n, 1 - \gamma\delta_n)$. The rest of the proof then follows verbatim from Theorem 3.1.

First, for (i), note that by Theorem 13.1 in Devroye et al. (2013), the VC dimension of the classifier $\tilde{\pi}(x,d) = \pi(x)(1-d)$ equals the VC dimension of $\pi(x)$, namely VC(Π). By Lemma 29.4 in Devroye et al. (2013) it follows that the VC dimension of $\tilde{\Pi}_n$ equals VC(Π).

Second, for (ii), for $\tilde{\pi}(x,d) = \pi(x)(1-d) + d$

$$P\Big(D_i = \tilde{\pi}(X_i, D_i)|Z_i, R_i = 1\Big) = \begin{cases} P(D_i = 1|Z_i, R_i = 1) & \text{if } \pi(X_i) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

It follows that $P\Big(D_i = \tilde{\pi}(X_i, D_i)|Z_i, R_i = 1\Big) \geq \min\{P(D_i = 1|Z_i, R_i = 1), P(D_i = 0|Z_i, R_i = 1)\} \in (\gamma, 1-\gamma)$. Similarly, I can show that $P\Big(D_i = \tilde{\pi}(X_i, D_i)|Z_i, R_i = 0, R_i^f = 1\Big) \in (\gamma, 1-\gamma)$ and $P(T_i = t|Z_i, R_i = 1, R_{k \in N_i}, Z_{k \in N_i}, |N_i|) \geq \delta_n$ almost surely for any $t \in \mathcal{T}_n$, under Assumption 2.2 (ii). Intuitively, because I *always* treat those units *also treated* in the experiment, overlap for $\tilde{\pi} \in \tilde{\Pi}_n$ is guaranteed, under overlap in the experiment. It follows that the propensity score $e_i(\tilde{\pi}) = e(\tilde{\pi}(X_i, D_i), T_i(\tilde{\pi}), Z_i, Z_{k \in N_i}, R_{k \in N_i}, |N_i|)$, $\tilde{\pi} \in \tilde{\Pi}_n$ satisfies the overlap conditions imposed in Assumption 2.2. Finally, it is easy to show that Lemma 2.1 directly holds also for any $\tilde{\pi} \in \tilde{\Pi}_n$, following verbatim the proof of Lemma 2.1, reweighting for $e_i(\tilde{\pi})$. The rest of the proof follows verbatim the one of Theorem 3.1.

D.4.3 Proof of Proposition B.2

Define $k(i)$ the partition $k \in \{1, \cdots, K^*\}$ associated with unit i under Algorithm 2 and $j(i)$ the fold j within partition $k(i)$ associated with i under Algorithm 2. Recall the definition of $\phi_s^m(i) = 1\{k(s) = k(i), j(s) \neq j(i)\}$ is Section B.2. Note that $\phi_s^m(i)$ are random variables since they depend on sampled indicators $R_1, \cdots, R_n$. By Lemma D.9, $K^* \leq \chi(A^2)$ almost surely.

For each partition k, Algorithm 2 creates J folds with the same number of units. I can write $\sum_{s=1}^n R_s \phi_s^m(i) \geq \Big\lfloor \frac{J-1}{J} \sum_{s=1}^n R_s 1\{k(s) = k(i)\} \Big\rfloor$ where I take the floor function for cases where J is not a multiple of the number of sampled units in the partition $k(i)$. We have

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^n \mathbb{E}\Big[\Big(1 + \sum_{s=1}^n R_s \phi_s^m(i)\Big)^{-2\zeta_m} | R_i = 1, A, Z\Big] \\ &\leq \frac{1}{n}\sum_{i=1}^n \mathbb{E}\Big[\max\Big\{1, \Big(\frac{J-1}{J}\sum_{s=1}^n R_s 1\{k(s) = k(i)\}\Big)^{-2\zeta_m}\Big\} | R_i = 1, A, Z\Big]. \end{aligned} \tag{D.38}$$

Worst-case partition Next, I replace the (random) partitions $k \in \{1, \cdots, K^*\}$ with *worst-case* non-random partitions. Denote $k^w(i) \in \{1, \cdots, \chi(A^2)\}$ the worst-case partition

$$\begin{aligned} &k^w(\cdot) \in \arg \max_{\underline{k}(i) \in \{1, \cdots, \chi(A^2)\}, i \in \{1, \cdots, n\}} \frac{1}{n}\sum_{i=1}^n \mathbb{E}\Big[\max\Big\{1, \Big(\frac{J-1}{J}\sum_{s=1}^n R_s 1\{\underline{k}(s) = \underline{k}(i)\}\Big)^{-2\zeta_m}\Big\} | R_i = 1, A, Z\Big] \\ &\quad \text{such that } \underline{k}(i) \neq \underline{k}(j), \forall j \in N_i \text{ or } N_i \cap N_j \neq \emptyset, \quad \sum_{k=1}^{\chi(A^2)} 1\{\underline{k}(i) = k\} = 1, \quad \forall i \in \{1, \cdots, n\}. \end{aligned} \tag{D.39}$$

Here, $k^w(\cdot)$ always exists by definition of $\chi(A^2)$.[18] In addition, k^w does not depend on the *realized* R by construction. I claim that

$$(\text{D.38}) \leq \frac{1}{n}\sum_{i=1}^n \underbrace{\mathbb{E}\Big[\max\Big\{1, \Big(\frac{J-1}{J}\sum_{s=1}^n R_s 1\{k^w(s) = k^w(i)\}\Big)^{-2\zeta_m}\Big\} | R_i = 1, A, Z\Big]}_{(I)} \tag{D.40}$$

Equation (D.40) holds for two reasons: (i) $K^* \leq \chi(A^2)$ by Lemma D.9; (ii) I can show that the constraint in Equation (D.39) is a stricter constraint than the constraint in Equation (24) for *any* realization of $(R_1, \cdots, R_n)$ (see the proof of Lemma D.9 for details).

[18] Existence is satisfied if a feasible solution to Equation (D.39) exists. One example is the smallest proper cover $\mathcal{C}_n(A^2)$ as in Definition D.1 for the adjacency matrix A^2. This satisfies the constraints in Equation (D.39) by definition. A proper cover always exists (e.g., if the network is fully connected, $\chi(A^2) = n$).

Upper bound on (I) Take any $i \in \{1, \cdots, n\}$ such that $1\{k^w(s) = k^w(i)\} = 1$ for some $s \neq i$. It follows from Cribari-Neto et al. (2000) (equation at the bottom of Page 274)

$$
\begin{aligned}
(I) &\leq (\frac{J-1}{J})^{-2\zeta_m}\mathbb{E}\Big[\Big(1 + \sum_{s\neq i} R_s 1\{k^w(s) = k^w(i)\}\Big)^{-2\zeta_m} | R_i = 1, A, Z\Big] \quad (\because R_i = 1) \\
&\leq \frac{(\frac{J-1}{J})^{-2\zeta_m}}{\Big(\frac{n_e}{n}\sum_{s\neq i} 1\{k^w(s) = k^w(i)\}\Big)^{2\zeta_m}} + \mathcal{O}\Big(\frac{1}{(\sum_{s\neq i} 1\{k^w(s) = k^w(i)\})^{2\zeta_m+1}}\Big) \qquad \text{(D.41)} \\
&\quad (\because n_e/n = \alpha \in (0,1), J = \mathcal{O}(1)).
\end{aligned}
$$

In the right-hand-side (first equation) we added one since $k^w(i) = k^w(s)$ for $s = i$. If instead there is no $s \neq i$, such that $1\{k^w(s) = k^w(i)\} = 1$, then trivially $(I) = \mathcal{O}(1)$.

Sum over all partitions Summing over all $\chi(A^2)$ partitions, we obtain

$$
\begin{aligned}
&\text{(D.40)} \leq \sum_{k=1}^{\chi(A^2)} \frac{\sum_{i=1}^n 1\{k^w(i) = k\}}{n}\mathbb{E}\Big[\max\Big\{1, \Big(\frac{J-1}{J}\sum_{s=1}^n R_s 1\{k^w(s) = k\}\Big)^{-2\zeta_m}\Big\}\Big] \leq \underbrace{\mathcal{O}(\chi(A^2)/n)}_{(A)} \\
&\underbrace{+\mathcal{O}\Big(\sum_{k=1}^{\chi(A^2)}\Big(\frac{\sum_{i=1}^n 1\{k^w(i) = k\}}{n}\Big)^{1-2\zeta_m}\Big(\frac{J}{(J-1)n_e}\Big)^{2\zeta_m}\Big) + \mathcal{O}\Big(\frac{1}{n}\sum_{k=1}^{\chi(A^2)}(1 + \sum_{s\neq i} 1\{k^w(i) = k\})^{-2\zeta_m}\Big)}_{(B)}
\end{aligned}
$$

where (B) correspond to cases where partitions $k^w(i)$ contain at least two elements (and bounded as in Equation (D.41))[19], and (A) corresponds to partitions with only one element, whose overall number is at most $\chi(A^2)$ (since there are at most $\chi(A^2)$ many partitions, and for such partitions $\frac{\sum_{i=1}^n 1\{k^w(i)=k\}}{n} = 1/n$). For (B) we write

$$
\begin{aligned}
(B) &\leq \mathcal{O}\Big(\chi(A^2)\Big(\frac{1}{\chi(A^2)}\sum_{k=1}^{\chi(A^2)}\frac{\sum_{i=1}^n 1\{k^w(i) = k\}}{n}\Big)^{1-2\zeta_m}\Big(\frac{J}{(J-1)n_e}\Big)^{2\zeta_m}\Big) \\
&\quad + \mathcal{O}\Big(\chi(A^2)\frac{1}{n}(\frac{1}{\chi(A^2)}\sum_{k=1}^{\chi(A^2)}\sum_{i=1}^n 1\{k(i) = k\})^{1-2\zeta_m}\Big) \quad (\because x^{-2\zeta_m} \leq x^{1-2\zeta_m} \text{ for } x \geq 1, \text{ concave } x^{1-2\zeta_m}).
\end{aligned}
$$

It follows that $(B) \leq \chi(A^2)\Big(\frac{J}{(J-1)n_e}\Big)^{2\zeta_m} + \mathcal{O}(\chi(A^2)n^{-2\zeta_m})$ $(\because \sum_{k=1}^{\chi(A^2)}\sum_{i=1}^n 1\{k(i) = k\} = n)$. From D.2, $\chi(A^2) \leq 2\mathcal{N}_n^2$, which completes the proof for the conditional mean after simple rearrangement (since the bound for (A) follows directly from Lemma D.2). The argument follows verbatim for $\mathcal{B}_n(A, Z)$, taking into account $1/\delta_n^2$, and omitted for brevity.

[19]For the first component in (A) we sum over all $i \in \{1, \cdots, n\}$ instead of $n-1$ elements since the last term is absorbed in $\mathcal{O}(1)$.

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